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Research & Education Association

THE HIGH SCHOOL GEOMETRY TUTOR®

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SUMMARY OF ESSENTIAL GEOMETRIC THEOREMS & PROPERTIES

AXIOMS

A quantity is equal to itself (reflexive law).

If two quantities are equal to the same quantity, they are equal to each other.

If a & b are any quantities, and a = b, then b = a (symmetric law).

The whole is equal to the sum of its parts.

If equal quantities are added to equal quantities, the sums are equal quantities.

If equal quantities are subtracted from equal quantities, the differences are equal quantities.

If equal quantities are multiplied by equal quantities, the products are equal quantities.

If equal quantities are divided by equal quantities (not 0), the quotients are equal quantities.

LINES

Two points determine one and only one line.

Let A & B be two points on a line. The set of points on the line between A & B and including A & B is called the line segment AB.

A line which divides a line segment into two segments with equal measure is called a bisector of the line segment. The intersection of the two is called the midpoint of the line segment.

A line segment has only one midpoint.

If three or more points lie on the same line, they are said to be collinear.

If two lines are cut by a transversal, nonadjacent angles on opposite sides of the transversal but on the interior of the two lines are called alternate interior angles.

If two lines are cut by a transversal, nonadjacent angles on opposite sides of the transversal and on the exterior of the two lines are called alternate exterior angles.

If two lines are cut by a transversal, angles on the same side of the transversal and in corresponding positions with respect to the lines are called corresponding angles.

TRIANGLES

A triangle is a closed three-sides figure.

The points of intersection of the sides of a triangle are called the vertices of the triangle.

A right triangle is a triangle with one right angle.

The side opposite the right angle in a right triangle is called the hypotenuse of the right triangle. The other two sides are called arms or legs of the right triangle.

An equilateral triangle is a triangle all of whose sides are of equal measure.

An isosceles triangle is a triangle that has at least two sides of equal measure. The third side is called the base of the triangle.

A scalene triangle is a triangle that has no pair of sides with equal measure.

The perimeter of a triangle is the sum of the measures of the sides of a triangle.

If one of the sides of a triangle is extended, the angle formed which is adjacent to an angle of the triangle is called an exterior angle.

An exterior angle of a triangle is equal to the sum of the nonadjacent interior angles.

Two triangles are congruent if, and only if, all of their corresponding parts are equal. ($\cong$)

If the hypotenuse and an acute angle of one right triangle are equal, respectively, to the hypotenuse and an acute angle of a second right triangle, the triangles are congruent.

If three sides of one triangle are equal, respectively, to three sides of a second triangle, the triangles are congruent. (SSS = SSS)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, the triangles are congruent. (SAS = SAS)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, the triangles are congruent. (ASA = ASA)

If two angles and any side of one triangle are equal, respectively, to two angles and any side of a second triangle, the triangles are congruent. (AAS = AAS)

The sum of the measures of the interior angles of a triangle is 180°.

If two angles of one triangle are equal respectively to two angles of a second triangle, their third angles are equal.

The acute angles of a right triangle are complementary.

A triangle can have at most one right or obtuse angle.

An acute triangle is a triangle with all acute angles.

An obtuse triangle is a triangle with one obtuse angle.

If the hypotenuse and an arm of one right triangle are equal, respectively, to the hypotenuse and an arm of a second right triangle, the right triangles are congruent.

The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base of the triangle.

If a triangle has two equal angles, then the sides opposite those angles are equal.

If two sides of a triangle are equal, then the angles opposite those sides are equal.

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

If a triangle has sides of length a, b, and c and $c^2 = a^2 + b^2$, then the triangle is a right triangle.

In a 30°-60° right triangle, the hypotenuse is twice the length of the side opposite the 30° angle. The side opposite the 60° angle is equal to the length of the side opposite the 30° angle multiplied by $\sqrt{3}$.

In an isosceles 45° right triangle, the hypotenuse is equal to the length of one of its arms multiplied by $\sqrt{2}$.

A line segment that connects the midpoints of two sides of a triangle is parallel to the third side and half as long.

A median of a triangle is a line segment drawn from one vertex of a triangle to the midpoint of the opposite side.

The medians of a triangle meet in a point.

An altitude of a triangle is a line segment drawn from one vertex of the triangle perpendicular to the opposite side, or if necessary, to an extension of the opposite side.

The altitudes of a triangle meet in a point.

The perpendicular bisectors of the sides of a triangle meet at a point which is equally distant from the vertices of the triangle.

The angle bisectors of a triangle meet in a point which is equidistant from the sides of the triangle.

The point of intersection of the medians of a triangle is 2/3 of the way from the vertex to the midpoint of the opposite side.

PARALLELISM

PARALLEL LINES:

Two lines are called parallel lines if, and only if, they are in the same plane and do not intersect.

Parallel lines are always the same distance apart.

Two lines are parallel if they are both perpendicular to a third line.

Given a line ℓ and a point P not on line ℓ, there is only one line through point P that is parallel to line ℓ.

If two lines are cut by a transversal so that alternate interior angles are equal, the lines are parallel.

If two parallel lines are cut by a transversal, all pairs of alternate interior angles are equal.

If two lines are cut by a transversal so that corresponding angles are equal, the lines are parallel.

If two parallel lines are cut by a transversal, each pair of corresponding angles is equal.

If two lines are cut by a transversal so that two interior angles on the same side of the transversal are supplementary, the lines are parallel.

If two parallel lines are cut by a transversal, pairs of interior angles on the same side of the transversal are supplementary.

If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

If two lines are cut by a transversal so that alternate interior angles are not equal, the lines are not parallel.

If two lines are cut by a transversal so that corresponding angles are not equal, the lines are not parallel.

If two lines are cut by a transversal so that two interior angles on the same side of the transversal are not supplementary, the lines are not parallel.

If two nonparallel lines are cut by a transversal, the pairs of alternate interior angles are not equal.

PARALLELOGRAM:

A parallelogram is a quadrilateral whose opposite sides are parallel. (▱)

Opposite sides of a parallelogram are equal.

Nonconsecutive angles of a parallelogram are equal.

Consecutive angles of a parallelogram are supplementary.

A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

The diagonals of a parallelogram bisect each other.

If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

If two opposite sides of a quadrilateral are both parallel and equal, the quadrilateral is a parallelogram.

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

RHOMBUS:

A rhombus is a parallelogram with two adjacent sides equal.

All sides of a rhombus are equal.

The diagonals of a rhombus are perpendicular to each other.

If the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.

The diagonals of a rhombus bisect the angles of the rhombus.

A square is a rhombus with a right angle.

RECTANGLE:

A rectangle is a parallelogram with one right angle.

All angles of a rectangle are right angles.

The diagonals of a rectangle are equal.

If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

TRAPEZOID:

A trapezoid is a quadrilateral with two and only two sides parallel. The parallel sides are called bases.

An isosceles trapezoid is a trapezoid whose nonparallel sides are equal. A pair of angles including one of the parallel sides is called a pair of base angles.

The base angles of an isosceles trapezoid are equal.

If a line joins the midpoints of two sides of a triangle, that line is parallel to and equal to one-half of the third side.

The median of a trapezoid is parallel to the bases and equal to one-half their sum.

The median of a trapezoid is the line joining the midpoints of the non-parallel sides.

If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on all transversals.

AREAS

The area of a rectangle is given by the formula $A = bh$ where b is the length of the base and h is the height of the rectangle.

The area of a triangle is given by the formula $A = 1/2\ bh$ where b is the length of a base and h is the corresponding height of the triangle.

The area of a parallelogram is given by the formula $A = bh$ where b is the length of a base and h is the corresponding height of the parallelogram. The side to which an altitude is drawn is called a base. The length of an altitude is called a height.

The area of a rhombus is equal to one-half the product of its diagonals.

The area of a trapezoid is given by the formula $A = 1/2\ h(b + b')$ where h is the height and b and b' are the lengths of the bases of the trapezoid.

The area of a circle is given by the formula $A = \pi r^2$ where π is approximately 3.14 and r is the radius of the circle.

Any two congruent figures have the same area.

RATIOS, PROPORTIONS, SIMILARITY

A ratio is the comparison of two numbers by their indicated quotient. A ratio is a fraction, with denominators not equal to zero.

A proportion is a statement that two ratios are equal.

In the proportion $a/b = c/d$ the numbers a & d are called the extremes of the proportion, and the numbers b & c are called the means of the proportion. The single term, d, is called the fourth proportional.

In a proportion, the product of the means is equal to the product of the extremes. (If $a/b = c/d$, then $ad = bc$.)

A proportion may be written by inversion. (If $a/b = c/d$, then $b/a = d/c$.)

The means may be interchanged in any proportion. (If $a/b = c/d$, then $a/c = b/d$.)

The extremes may be interchanged in any proportion. (If $a/b = c/d$, then $d/b = c/a$.)

A proportion may be written by addition. (If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.)

A proportion may be written by subtraction. (If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.)

If three terms of one proportion are equal, respectively, to three terms of a second proportion, the fourth terms are equal. Thus, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b}$.

A line parallel to one side of a triangle divides the other two sides proportionally.

If a line divides two sides of a triangle proportionally, it is parallel to the third side.

The bisector of one angle of a triangle divides the opposite side in the same ratio as the other two sides.

All congruent triangles are similar.

Two triangles similar to the same triangle are similar to each other.

If two triangles have one angle of one equal to one angle of the other and the respective sides including these angles are in proportion, the triangles are similar.

If three sides of one triangle are in proportion to the three corresponding sides of a second triangle, the triangles are similar.

Two triangles are similar if, and only if, two angles of one triangle are equal to two angles of the other triangle.

The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

Two polygons are similar if, and only if, all pairs of corresponding angles are equal and all pairs of corresponding sides are in proportion.

CIRCLES

A circle is a set of points in the same plane equidistant from a fixed point called its center. A line segment drawn from the center of the circle to one of the points on the circle is called a radius of the circle.

The length of a diameter is twice the length of a radius.

A portion of a circle is called an arc of the circle.

In a circle, parallel lines intercept equal arcs.

The circumference of a circle is the distance about a circle. The circumference of a circle is given by the formula, $C = 2\pi r$ where r is the radius of the circle.

Two circles are congruent if, and only if, their radii or diameters are equal.

A semicircle is an arc of a circle whose endpoints lie on the extremities of a diameter of the circle.

An arc greater than a semicircle is called a major arc.

An arc less than a semicircle is called a minor arc.

An angle whose vertex is at the center of a circle and whose sides are radii is called a central angle.

The number of degrees in the arc intercepted by a central angle is equal to the number of degrees in the central angle. This number is called the measure of the arc.

A sector of a circle is the set of points between two radii and their intercepted arc.

The line passing through the centers of two circles is called the line of centers.

A line that intersects a circle in two points is called a secant.

A line segment joining two points on a circle is called a chord of the circle.

A chord that passes through the center of the circle is called a diameter of the circle.

An angle whose vertex is on the circle and whose sides are chords of the circle is called an inscribed angle.

The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.

An angle formed by a tangent and a chord is equal to one-half the measure of the intercepted arc.

If two chords intersect within a circle, each angle formed is equal to one-half the sum of its intercepted arc and the intercepted arc of its vertical angle.

An angle formed by the intersection of two secants outside a circle is equal to one-half the difference of its intercepted arcs.

An angle formed by the intersection of a tangent and a secant outside a circle is equal to one-half the difference of the intercepted arcs.

A line drawn from the center of a circle perpendicular to a chord bisects the chord and its arc.

A line drawn from the center of a circle to the midpoint of a chord is perpendicular to the chord.

The perpendicular bisector of a chord passes through the center of the circle.

In the same circle or congruent circles equal chords have equal arcs.

In the same circle or congruent circles equal arcs have equal chords.

In the same circle or congruent circles, equal chords are equidistant from the center.

In the same circle or congruent circles, chords equidistant from the center are equal.

If two chords intersect within a circle, the product of the segments of one chord is equal to the product of the segments of the other chord.

If two circles intersect in two points, their line of centers is the perpendicular bisector of their common chord.

A line that has one and only one point of intersection with a circle is called a tangent to the circle. Their common point is called a point of tangency.

If two tangents are drawn to a circle from a point outside the circle, the tangents are of equal measure.

A line drawn from a center of a circle to a point of tangency is perpendicular to the tangent passing through the point of tangency. Also, if a line is perpendicular to a radius at its intersection with its circle, the line is tangent to the circle.

If two circles are tangent either internally or externally, the point of tangency and the centers of the circles are collinear.

If a secant and a tangent are drawn to a circle, the measure of the tangent is the mean proportional between the secant and its external segment.

If two secants are drawn to a circle from a point outside the circle, the products of the secants and their external segments are equal.

The opposite angles of an inscribed quadrilateral are supplementary.

If a parallelogram is inscribed within a circle, it is a rectangle.

POLYGONS

A polygon is a convex figure with the same number of sides as angles.

A quadrilateral is a polygon with four sides.

An equilateral polygon is a polygon all of whose sides are of equal measure.

An equiangular polygon is a polygon all of whose angles are of equal measure.

A regular polygon is a polygon that is both equilateral and equiangular.

A circle can be inscribed in any regular polygon.

A circle can be circumscribed about any regular polygon.

The center of a circle that is circumscribed about a regular polygon is the center of the circle that is inscribed within the regular polygon.

If a circle is divided into n equal arcs, the chords of the arcs will form a regular polygon. $(n > 2)$

If a circle is divides into n equal arcs and tangents are drawn to the circle at the endpoints of these arcs, the figure formed by the tangents will be a regular polygon. $(n > 2)$

An apothem of a regular polygon bisects its respective side.

The radius of a regular polygon is a line segment drawn from the center of the polygon to one of its vertices.

A central angle of a regular polygon is the angle formed by radii drawn to two consecutive vertices.

A central angle of a regular polygon is given by the formula, $x = \frac{2}{n} . 180°$, where n is the number of sides of the polygon.

The area of a regular polygon is equal to the product of one-half its apothem and its perimeter.

The sum of the interior angles of a polygon is given by the formula $S = (n - 2) \cdot 180°$, where n is the number of sides of the polygon.

The measure of an interior angle, a, of a regular polygon is given by the formula: $a = \frac{(n - 2) \cdot 180°}{n}$ where n is the number of sides of of the polygon.

The sum of the exterior angles of a polygon is always 360°.

The area of a sector of a circle divided by the area of the circle is equal to the measure of its central angle divided by 360°.

INEQUALITIES

If the same quantity is added to both sides of an inequality, the sums are unequal and in the same order. (If $a < b$, then $a + c < b + c$.)

If equal quantities are added to unequal quantities, the sums are unequal and in the same order. (If $a < b$ and $c = d$, then $a + c < b + d$.)

If unequal quantities are subtracted from equal quantities, the differences are unequal and in the opposite order. (If $a<b$ and $c = d$, then $c - a>d - b$.)

If both sides of an inequality are multiplied by a positive number, the products are unequal and in the same order. (If $a<b$ and c is a positive number, then $ac<bc$.)

If both sides of an inequality are multiplied by a negative number, the products are unequal in the opposite order. (If $a<b$ and c is a negative number, then $ac>bc$.)

The relation "<" is transitive. (If $a<b$ and $b<c$, then $a<c$.)

If unequal quantities are added to unequal quantities of the same order, the sums are unequal quantities and in the same order. (If $a<b$ and $c<d$, then $a + c<b + d$.)

In a triangle an exterior angle is greater than either nonadjacent interior angle.

If two sides of a triangle are unequal, the angles opposite these sides are unequal in the same order.

If two angles of a triangle are unequal, the sides opposite these angles are unequal in the same order.

If two sides of one triangle are equal to two sides of a second triangle and the included angle of the first is greater than the included angle of the second, then the third side of the first triangle is greater than the third side of the second.

If two sides of one triangle are equal to two sides of a second triangle and the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.

In the same circle or in equal circles, the greater of two central angles will intercept the greater arc.

In the same circle or in equal circles, the greater of two arcs will be intercepted by the greater of the central angles.

In the same circle or equal circles, the greater of two chords intercepts the greater minor arc.

In the same circle or equal circles, the greater of two minor arcs has the greater chord.

If two unequal chords form an inscribed angle within a circle, the shorter chord is the farther from the center of the circle.

LOCUS

A circle is the locus of points at a given distance from a fixed point, called its center, and all in the same plane.

The locus of points whose coordinates satisfy a given linear equation is a straight line.

The locus of the vertex of the right angle of a right triangle with a fixed hypotenuse is a circle with the hypotenuse as diameter.

The locus of all points equidistant from the sides of an angle is the angle bisector.

The locus of all points equidistant from two points is the perpendicular bisector of the segment joining the two points.

The locus of a point at a given distance from a given line is a pair of lines, one on each side of the given line, parallel to the given line and at the given distance from it.

COORDINATE GEOMETRY

Directed distance of a horizontal line segment from one point to a second is the x-coordinate of the second minus the x-coordinate of the first.

Directed distance of a vertical line segment from one point to a second is the y-coordinate of the second minus the y-coordinate of the first.

The slope, m, of a nonvertical line segment determined by points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

On a nonvertical line, all segments of the line have the same slope.

Two nonvertical lines are perpendicular if, and only if, their slopes are negative reciprocals.

Two nonvertical lines are parallel if, and only if, they have the same slope. Vertical lines are parallel.

A linear equation is any equation which can be written in the form of $Ax + By + C = 0$.

CHAPTER 1

METHODS OF PROOF

DEDUCTIVE REASONING

• PROBLEM 1-1

How may the following three statements be arranged so that the first two will make it possible to deduce the third? (a) An eagle has feathers. (b) All birds have feathers. (c) An eagle is a bird.

Solution: An arrangement of statements that would allow you to deduce the third one from the preceeding two is called a syllogism. A syllogism has three parts.

The first part is a general statement concerning a whole group. This is called the major premise.

The second part is a specific statement which indicates that a certain individual is a member of that group. This is called the minor premise.

The third and final part of a syllogism is a statement to the effect that the general statement concerning the group also applies to the individual. This statement is called a deduction.

The technique of employing a syllogism to arrive at a conclusion is called deductive reasoning. This technique will be used in this problem.

Statement (b) "All birds have feathers" is the major premise since it assigns a general characteristic to the whole group, in this case birds.

Statement (c) "An eagle is a bird" links the eagle to the larger group mentioned in the major premise and, therefore, (c) is the minor premise.

Statement (a) "An eagle has feathers" assigns the general trait, feathers, to the individual, eagles, and, thus, qualifies as the deduction.

Therefore, the order of the statements that allows you to deduce the third from the first two is (b), (c), (a).

• PROBLEM 1-2

All residents of this state who are registered voters are 18 years of age or older. If John is a resident of this state, by valid reasoning which of the following can we conclude: (a) If John is 18 or over, he is a registered voter; (b) If John is a registered voter, he is 18 or over. (c) If John is not a registered voter, he is not 18 or over.

Solution: To find the logically correct statement, we find which of the choices forms a valid syllogism with the given statement. The given statement "All residents of this state who are registered voters are 18 years of age or older" is the major premise, because it attributes a characteristic ("being 18 or older") to a general group ("residents of this state who are registered voters"). A minor premise links a particular example, John, to the general group. Note that there are two conditions to being a member of the group: a member must (1) be a resident of the state; and (2) be a registered voter. Therefore, any valid minor premise must include both conditions.

One condition of the minor premise is common to all the choices and therefore written in the front - "If John is a resident of this state". The second condition of the minor premise is different in each choice.

For choice (a), the minor premise is "John is a resident of this state and John is 18 or over". This is not enough to make John a resident of the state who is a registered voter. Thus, John is not linked to the group and choice (a) is not valid reasoning. Thinking about this in another way, it is definitely possible that apathetic John is over 18, and a resident of the state; and yet John may not have bothered to register.

For choice (b), the minor premise is "John is a resident of this state and John is a registered voter." This is enough to qualify John as a member of the group. Thus, the quality of the group can be linked with John in the conclusion - John is 18 or over. This is, indeed, the conclusion of (b). Thus choice (b) leads to valid reasoning.

For choice (c), the minor premise is "John is a resident of this state and John is not a registered voter." This is not enough to make John a member of the general group. The reasoning is invalid. It is possible for an apathetic 50 year old citizen not to register to vote.

• PROBLEM 1-3

Identify the hypothesis and conclusion in the statement, "The median to the base of an isosceles triangle is perpendicular to the base."

Solution: The statement dealt with in this problem is a "simple" sentence, as opposed to the "if-then" type of statement. In general, if a statement is written as a simple sentence, then the subject of the sentence is the hypothesis, and the predicate is the conclusion.

In this example, the phrase "The median to the base of an isosceles triangle" is the subject of the sentence and, consequently, the hypothesis is "a line segment is the median to the base of an isosceles triangle." The phrase "is perpendicular to the base" is the predicate and, therefore, the conclusion is "the line is perpendicular to the base."

• PROBLEM 1-4

State whether the condition in the hypothesis is: both necessary and sufficient, necessary but not sufficient, not necessary but sufficient, neither necessary nor sufficient. (a) If a man is sick, he is a patient in a hospital. (b) If a man is a senator, he is a member of Congress. (c) If a figure is a square, then the figure is a quadrilateral.

Solution: A necessary condition is a fact or set of facts which must be given in the hypothesis of a conditional statement to make it possible to deduce the conclusion.

A sufficient condition is a fact or set of facts which, when given in the hypothesis, supplies enough information to make it possible to deduce the conclusion.

If a condition is necessary, it is essential that the fact be given in the hypothesis to have any chance of deducing the conclusion. A necessary condition will not always be enough to allow one to arrive at a conclusion by formal logic. On the other hand, a sufficient condition, when given, will always lead to a logical deduction of a certain conclusion.

(a) In this statement the hypothesis is "a man is sick," the conclusion is "he is a patient in a hospital." Certainly, if a man is a patient in a hospital, he must be sick. Therefore, it is at least necessary that he be sick before he can be in the hospital.

However, it is not sufficient that he be merely sick as the sole prerequisite for being in the hospital because one has many ways of being sick and not ending up in the hospital.

We conclude then that the hypothesis is necessary but not sufficient to deduce the stated conclusion.

In this part we showed the converse of that statement to be true which is a standard way to decide if the hypothesis is necessary. To prove sufficiency we must prove that the statement is true.

(b) The hypothesis is "the man is a senator," while the conclusion is "he is a member of Congress."

The converse is "If he is a member of Congress, he is a senator." This is false, because members of the House of Representatives are members of Congress but certainly not senators. Therefore, in the original statement, the hypothesis is not necessary because the conclusion can be reached under much different hypothetical conditions, i.e. if the hypothesis were "he is a member of the House of Representatives."

The statement is true and this lets us conclude that the hypothesis is sufficient for the conclusion. Once a person is a senator he is automatically a member of Congress.

Therefore, the hypothesis is not necessary but sufficient for the stated conclusion.

(c) In this statement, the hypothesis is "a figure is a square" and the conclusion is "the figure is a quadrilateral."

The statement is true since every square is a four sided polygon and every four sided polygon is a quadrilateral. Thus, the hypothetical condition is sufficient.

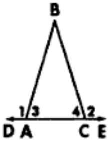
The converse is "if the figure is a quadrilateral, then the figure is a square." This is false since the figure could also be a rectangle or trapezoid. Since the converse is false the hypothesis of the original statement is not necessary.

The hypothesis is sufficient but not necessary.

DIRECT PROOF

• PROBLEM 1-5

In the figure shown, the measure of ∡ DAC equals the measure of ∡ ECA and the measure of ∡1 equals the measure of ∡2. Show that the measure of ∡3 equals the measure of ∡4.



Solution: This proof will require the subtraction postulate, which states that if equal quantities are subtracted from equal quantities, the differences are equal.

GIVEN: $\angle DAC \cong \angle ECA$, $\angle 1 \cong \angle 2$

PROVE: $\angle 3 \cong \angle 4$

STATEMENTS	REASONS
1. $m\angle DAC = m\angle ECA$ $m\angle 1 = m\angle 2$	1. Given.
2. $m\angle DAC - m\angle 1 = m\angle ECA - m\angle 2$	2. Subtraction Postulate.
3. $m\angle 3 = m\angle 4$	3. Substitution Postulate.

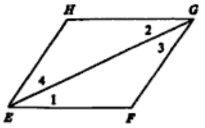
• PROBLEM 1-6

Every line in a proof requires justification. In the proof below the reasons have been omitted. Give the reasons.

GIVEN: $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$

PROVE: $\angle H \cong \angle F$

STATEMENTS	REASONS
1. $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	1.
2. $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$	2.
3. $m\angle 1 + m\angle 3 + m\angle F = 180$ and $m\angle 2 + m\angle 4 + m\angle H = 180$	3.
4. $m\angle 1 + m\angle 3 + m\angle F = m\angle 2 + m\angle 4 + m\angle H$	4.
5. $m\angle F = m\angle H$	5.



Solution: For any statement, several reasons may be possible. In steps 2, 3 and 4, below, alternative reasons are given. Any one will do. Even for a given reason, there may be a choice between citing or writing out the theorem. Generally, the shorter the reason, the better. Postulates and famous theorems can be named; however, theorems that are less well known should be written out.

The reasons for the proof in this question are:

(1) Given.

(2) The Angle Addition Postulate or the Addition Property of Equality; if a = b and c = d, then a + c = b + d.

(3) The measures of the interior angles of a triangle sum to 180° degrees or the interior angles of a triangle form a linear pair.

(4) Transitivity Postulate or Substitition Postulate or if a = b and b = c, then a = c.

(5) Subtraction Property of Equality.

INDIRECT PROOF

• PROBLEM 1-7

Prove, by indirect method, that if two angles are not congruent, then they are not both right angles.

Solution: Indirect proofs involve considering two possible outcomes, the result we would like to prove and its negative, and then showing, under the given hypothesis, that a contradiction of prior known theorems, postulates, or definitions is reached when the negative is assumed.

In this case the outcomes can be that the two angles are not right angles or that the two angles are right angles. Assume the negative of what we want to prove - that the two angles are right angles.

The given hypothesis in this problem is that the two angles are not congruent. A previous theorem states that all right angles are congruent. Therefore, the conclusion we have assumed true leads to a logical contradiction. As such, the alternative conclusion must be true. Therefore, if two angles are not congruent, then they are not both right angles.

• PROBLEM 1-8

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution: Mathematical induction is a method of proof. The steps are:
(1) The verification of the proposed formula or theorem for the smallest value of n. It is desirable, but not necessary, to verify it for several values of n.
(2) The proof that if the proposed formula or theorem is true for $n = k$, some positive integer, it is true also for $n = k+1$. That is, if the proposition is true for any particular value of n, it must be true for the next larger value of n.
(3) A conclusion that the proposed formula holds true for all values of n.

Proof: Step 1. Verify:

For $n = 1$: $1^2 = \frac{1}{6}(1)(1+1)[2(1)+1] = \frac{1}{6}(1)(2)(3) = \frac{1}{6}(6) = 1$

$$1 = 1$$

For $n = 2$: $1^2 + 2^2 = \frac{1}{6}(2)(2+1)[2(2)+1] = \frac{1}{6}(2)(3)(5) = \frac{1}{6}(6)(5)$

$$1 + 4 = (1)(5)$$

$$5 = 5$$

For $n = 3$: $1^2 + 2^2 + 3^2 = \frac{1}{6}(3)(3+1)[2(3)+1]$

$$1 + 4 + 9 = \frac{1}{6}(3)(4)(7) = \frac{1}{6}(12)(7) = 14$$

$$14 = 14$$

Step 2. Let k represent any particular value of n. For $n = k$, the formula becomes

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1). \qquad (A)$$

For $n = k+1$, the formula is

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{6}(k+1)[(k+1) + 1][2(k+1) + 1]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3). \qquad (B)$$

We must show that if the formula is true for $n = k$, then it must be true for $n = k+1$. In other words, we must show that (B) follows from (A). The left side of (A) can be converted into the left side of (B) by merely adding $(k+1)^2$. All that remains to be demonstrated is that when $(k+1)^2$ is added to the right side of (A), the result is the right side of (B).

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

Factor out $(k+1)$:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + \frac{(k+1)6}{6}\right]$$

$$= (k+1)\,\frac{2k^2 + k + 6k + 6}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3),$$

since $\quad 2k^2 + 7k + 6 = (k+2)(2k+3).$

Thus, we have shown that if we add $(k+1)^2$ to both sides of the equation for $n = k$, then we obtain the equation or formula for $n = k+1$. We have thus established that if (A) is true, then (B) must be true; that is, if the formula is true for $n = k$, then it must be true for $n = k+1$. In other words, we have proved that if the proposition is true for a certain positive integer k, then it is also true for the next greater integer $k+1$.

Step 3. The proposition is true for $n = 1,2,3$ (Step 1). Since it is true for $n = 3$, it is true for $n = 4$ (Step 2, where $k = 3$ and $k+1 = 4$). Since it is true for $n = 4$, it is true for $n = 5$, and so on, for all positive integers n.

• PROBLEM 1-9

Prove by mathematical induction that

$$1 + 7 + 13 + \ldots + (6n - 5) = n(3n - 2).$$

Solution: (1) The proposed formula is true for $n = 1$, since $1 = 1(3 - 2)$.

(2) Assume the formula to be true for $n = k$, a positive integer; that is, assume

(A) $\quad 1 + 7 + 13 + \ldots + (6k - 5) = k(3k - 2).$

Under this assumption we wish to show that

(B) $\quad 1 + 7 + 13 + \ldots + (6k - 5) + (6k + 1) = (k+1)(3k+1).$

When $(6k+1)$ is added to both members of (A), we have on the right

$$k(3k-2) + (6k+1) = 3k^2 + 4k + 1 = (k+1)(3k+1);$$

hence, if the formula is true for $n = k$ it is true for $n = k + 1$.

(3) Since the formula is true for $n = k = 1$ (Step 1), it is true for $n = k + 1 = 2$; being true for $n = k = 2$ it is true for $n = k + 1 = 3$; and so on, for every positive integral value of n.

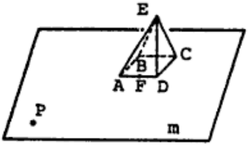
CHAPTER 2

BASIC ELEMENTS

LINES

• PROBLEM 2-1

Using the accompanying three-dimensional model, determine whether each of the following sets is collinear and/or coplanar: {A,F,D}, {P,A,F}, {A,B,C,E}, and {A,F,D,E}.



Solution: A set of points is collinear if there exists a straight line that contains every point in the set. A set of points is coplanar if there exists a plane that contains every point in the set.

Each point in the set {A,F,D} lie on the line $\overleftrightarrow{AD}$. Therefore, the set is collinear. Furthermore, points on the same line automatically lie on the same plane. Thus, the set of points {A,F,D} is both collinear and coplanar.

Each point in the set {P,A,F} is contained in plane m and, as such, points P, A, and F are coplanar. However, point P does not lie on the line determined by F and P, $\overleftrightarrow{FP}$; and, hence, the points are not collinear.

{A,B,C,E} are neither collinear nor coplanar. Point E does not lie on the plane determined by points A, B, and C (plane m) - thus the set cannot be coplanar. Points E and

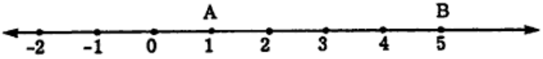
C do not lie on the line determined by points A and B - thus the set cannot be collinear.

The set of points {A,F,D,E} is not collinear. Although F lies on the line determined by points A and D, point E does not. Therefore, the set is noncollinear. To determine whether or not the set is coplanar, consider the plane that contains points A, E, and D. Therefore, the set is coplanar if point F lies on this plane.

Since points A and D are contained in this new plane, the line determined by these points, $\overleftrightarrow{AD}$, and every point on the line lies on the new plane. Since F is a point on $\overleftrightarrow{AD}$, point F lies on the new plane. Therefore, points A,F, D, and E are coplanar.

• PROBLEM 2-2

Find point C between A and B in the figure below such that $\overline{AC} \cong \overline{CB}$.



Solution: We must determine point C in such a way that $\overline{AC} \cong \overline{CB}$, or $AC = CB$. We are first given that C is between A and B. Therefore, since the measure of the whole is equal to the sum of the measure of its parts:

(I) $AC + CB = AB$

Using these two facts, we can find the length of AC. From that we can find C.

First, since $AC = CB$, we substitute AC for CB in equation (I)

(II) $AC + AC = AB$

(III) $2(AC) = AB$

Dividing by 2 we have

(IV) $AC = \frac{1}{2} AB$

To find AC, we must know AB. We can find AB from the coordinates of A and B. They are 1 and 5, respectively. Accordingly,

(V) $AB = |5 - 1|$

(VI) $AB = 4$

We substitute 4 for AB in equation (IV)

(VII) $AC = \frac{1}{2}(4)$

(VIII) $AC = 2.$

Therefore, C is 2 units from A. Since C is between A and B, the coordinate of C must be 3.

ANGLES

• PROBLEM 2-3

In the following quadrilateral, find the angles from vertices A and B that are subtended by side $\overline{CD}$.

Solution: To do this problem, we first must define "subtend." Consider a point X and a line segment $\overline{YZ}$. Now, draw rays from the point through the endpoints of the segment (see figure.) There are two ways of defining the relation between the angle $\sphericalangle YXZ$ and segment $\overline{YZ}$. The angle $\sphericalangle YXZ$ "intercepts" segment $\overline{YZ}$. From another point of view, the segment $\overline{YZ}$ "subtends" $\sphericalangle YXZ$.

In this problem, we are given that $\overline{DC}$ is the segment. We are asked first to consider A as the outside point, and then B. If A is the outside point, and $\overline{CD}$ is the segment, then $\overline{CD}$ subtends $\sphericalangle DAC$. If B is the outside point, and $\overline{CD}$ is the segment, then $\overline{CD}$ subtends $\sphericalangle DBC$.

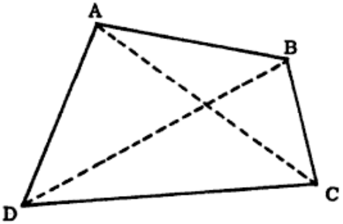


Figure 1

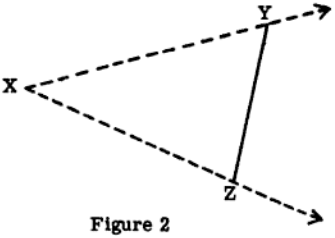
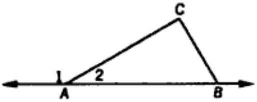


Figure 2

• PROBLEM 2-4

In the figure, we are given $\overleftrightarrow{AB}$ and triangle ABC. We are told that the measure of $\sphericalangle 1$ is five times the measure of $\sphericalangle 2$. Determine the measures of $\sphericalangle 1$ and $\sphericalangle 2$.



Solution: Since ∢1 and ∢2 are adjacent angles whose non-common sides lie on a straight line, they are, by definition, supplementary. As supplements, their measures must sum to 180°.

If we let x = the measure of ∢2
then, $5x$ = the measure of ∢1.

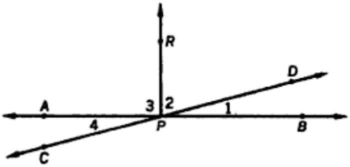
To determine the respective angle measures, set $x + 5x = 180$ and solve for x. $6x = 180$. Therefore, $x = 30$ and $5x = 150$.

Therefore, the measure of ∢1 = 150 and the measure of ∢2 = 30.

PERPENDICULARITY

• PROBLEM 2-5

We are given straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at point P. $\overleftrightarrow{PR} \perp \overleftrightarrow{AB}$ and the measure of ∢APD is 170. Find the measures of ∢1, ∢2, ∢3, and ∢4.



Solution: This problem will involve making use of several of the properties of supplementary and vertical angles, as well as perpendicular lines.

∢APD and ∢1 are adjacent angles whose non-common sides lie on a straight line, $\overleftrightarrow{AB}$. Therefore, they are supplements and their measures sum to 180°.

$$m\measuredangle APD + m\measuredangle 1 = 180.$$

We know $m\measuredangle APD = 170$. Therefore, by substitution, $170 + m\measuredangle 1 = 180$. This implies $m\measuredangle 1 = 10$.

∢1 and ∢4 are vertical angles because they are formed by the intersection of two straight lines, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$, and their sides form two pairs of opposite rays. As vertical angles, they are, by theorem, of equal measure. Since $m\angle 1 = 10$, then $m\angle 4 = 10$.

Since $\overleftrightarrow{PR} \perp \overleftrightarrow{AB}$, at their intersection the angles formed must be right angles. Therefore, $\angle 3$ is a right angle and its measure is 90. $m\angle 3 = 90$.

The figure shows us that $\angle APD$ is composed of $\angle 3$ and $\angle 2$. Since the measure of the whole must be equal to the sum of the measures of its parts, $m\angle APD = m\angle 3 + m\angle 2$. We know the $m\angle APD = 170$ and $m\angle 3 = 90$, therefore, by substitution, we can solve for $m\angle 2$, our last unknown.

$$170 = 90 + m\angle 2$$

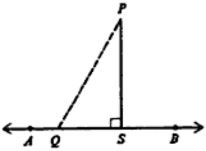
$$80 = m\angle 2$$

Therefore, $m\angle 1 = 10$, $m\angle 2 = 80$

$m\angle 3 = 90$, $m\angle 4 = 10$.

• PROBLEM 2-6

Show that the shortest segment joining a line with an external point is the perpendicular segment from the point to the line.



Solution: To prove the statement, choose any other line segment joining the point and the line; and show that it must be of greater length than the perpendicular. Note that the new segment, the perpendicular, and the given line form a right triangle with the right angle opposite the new segment. Since the right angle is the largest angle of the triangle, the side opposite it, the new segment, must be greater than either of the other sides. In particular, the new segment is longer than the perpendicular. Since the new segment was any possible segment, the perpendicular is thus shown to be shorter than any other segment.

GIVEN: $\overleftrightarrow{AB}$ with the external point P; $\overleftrightarrow{PS} \perp \overleftrightarrow{AB}$; $\overline{PQ}$ is any segment from P to $\overleftrightarrow{AB}$ other than the perpendicular from P to $\overleftrightarrow{AB}$.

PROVE: PS < PQ

STATEMENT	REASONS
1. $\overleftrightarrow{AB}$ with the external point P; $\overleftrightarrow{PS} \perp \overleftrightarrow{AB}$	1. Given
2. $\measuredangle$ PSA is a right angle	2. Perpendicular lines form right angles.
3. Select any point Q on $\overleftrightarrow{AB}$	3. Point Uniqueness Postulate.
4. $\measuredangle$ PQS is acute	4. If a triangle has one right angle, then its other two angles must be acute.
5. m $\measuredangle$ PQS < m $\measuredangle$ PSA	5. Definition of an acute angle.
6. PS < PQ	6. If two angles of a triangle are not congruent, then the sides opposite those angles are not congruent, the longer side being opposite the angle with the greater measure.

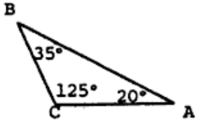
CHAPTER 3

CONGRUENCE

TRIANGLES

• PROBLEM 3-1

In Δ ABC, $m \sphericalangle C = 125^\circ$ and $m \sphericalangle B = 35^\circ$. As drawn in the figure, which is the shortest side of the triangle?



Solution: When the angles of a triangle are of unequal measure, the sides are, correspondingly, of unequal length - that is, the shortest side will be opposite the smallest angle.

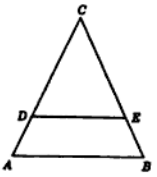
Since the sum of the measures of the angles of a triangle is 180°, and since $m \sphericalangle B = 35^\circ$ and $m \sphericalangle C = 125^\circ$, the $m \sphericalangle A = 180^\circ - (35^\circ + 125^\circ) = 180^\circ - 160^\circ = 20^\circ$. Therefore, the side opposite $< A$ will be the shortest. $\overline{BC}$ is this side.

Therefore, $\overline{BC}$ is the shortest side of Δ ABC.

• PROBLEM 3-2

Given: $m \sphericalangle A = m \sphericalangle B$. AD = BE .
Prove: $m \sphericalangle CDE = m \sphericalangle CED$.

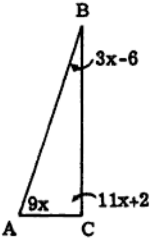
Solution: This proof will revolve around the theorem stating that two angles of a triangle are of equal measure if and only if the sides opposite them are of equal length.



Statements	Reasons
1. $m\angle A = m\angle B$	1. Given
2. $CA = CB$	2. If a triangle has two angles of equal measure, then the sides opposite those angles are equal in length.
3. $CB = CE + BE$ $CA = CD + AD$	3. The measure of the whole is equal to the sum of the measures of its parts.
4. $CD + AD = CE + BE$	4. A quantity may be substituted for its equal.
5. $AD = BE$	5. Given
6. $CD = CE$	6. If equal quantities are subtracted from equal quantities, the differences are equal quantities.
7. $m\angle CDE = m\angle CED$	7. If two sides of a triangle are equal, then the angles opposite those sides are equal.

• PROBLEM 3-3

As seen in the accompanying diagram, ΔABC is constructed in such a way that the measure of $\angle A$ equals $9x$, $m\angle B$ equals $3x - 6$, and $m\angle C$ equals $11x + 2$, x being some unknown. Show that ΔABC is a right triangle.



Solution: A triangle is a right triangle if one of its angles is a right angle. The best way to determine the "rightness" of this triangle would be to sum the measures of all its angles, set this sum equal to 180°, and solve for the unknown x. If the measure of one angle turns out to be 90°, then it is a right angle and the triangle is a right triangle. The algebra is as follows:

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180 \\ 9x + 3x - 6 + 11x + 2 &= 180 \\ 23x - 4 &= 180 \\ 23x &= 184 \\ x &= 8 \end{aligned}$$

Therefore,

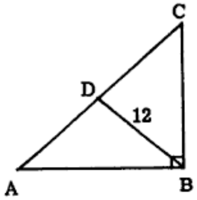
$\angle A$ measures $(9)(8)$ or 72
$\angle B$ measures $(3)(8) - 6$ or 18

and $\angle C$ measures $(11)(8) + 2$ or 90

Therefore, since $\angle C$ measures 90°, ΔABC is a right triangle.

• PROBLEM 3-4

The length of the median drawn to the hypotenuse of a right triangle is 12 inches. Find the length of the hypotenuse. (See figure).



Solution: A theorem tells us that the length of the median to the hypotenuse of a right triangle is equal to one-half the length of the hypotenuse. We must identify the median, the hypotenuse, their respective lengths, and substitute them according to the rule cited, and to solve for any unknowns.

$\overline{AC}$ is the hypotenuse, $\overline{BD}$ the median of length 12", and the length of $\overline{AC}$ is unknown. By applying the above theorem we know,

(1). $BD = \frac{1}{2}AC$ and, by, substitution

(2). $12'' = \frac{1}{2}AC$ which implies that $AC = 24''$.

Therefore, the length of hypotenuse $\overline{AC}$ is 24 in.

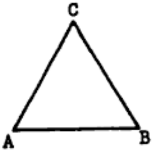
• PROBLEM 3-5

Prove that an equilateral triangle has three equal angles.

Solution: Draw equilateral ΔABC.

Hence, the problem can be restated as:

Given: equilateral ΔABC



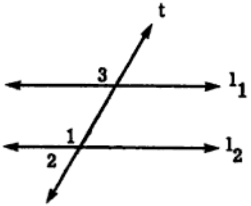
Prove: $m\angle A = m\angle B = m\angle C$.

STATEMENTS	REASONS
1. equilateral $\triangle ABC$	1. Given.
2. $AC \cong BC \cong AB$	2. Definition of an equilateral triangle.
3. $\angle A \cong \angle B$ $\angle A \cong \angle C$ $\angle B \cong \angle C$	3. If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
4. $\angle A \cong \angle B \cong \angle C$	4. Transitive Property of Congruence.

PARALLEL LINES

• PROBLEM 3-6

Given: $\angle 2$ is supplementary to $\angle 3$.
Prove: $\ell_1 \parallel \ell_2$.

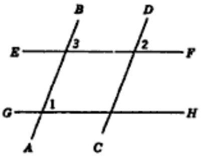


Solution: Given two lines intercepted by a transversal, if a pair of corresponding angles are congruent, then the two lines are parallel. In this problem, we will show that since $\angle 1$ and $\angle 2$ are supplementary and $\angle 2$ and $\angle 3$ are supplementary, $\angle 1$ and $\angle 3$ are congruent. Since corresponding angles $\angle 1$ and $\angle 3$ are congruent, it follows $\ell_1 \parallel \ell_2$

Statement	Reason
1. $\measuredangle 2$ is supplementary to $\measuredangle 3$	1. Given
2. $\measuredangle 1$ is supplementary to $\measuredangle 2$	2. Two angles that form a linear pair are supplementary.
3. $\measuredangle 1 \cong \measuredangle 3$	3. Angles supplementary to the same angle are congruent.
4. $\ell_1 \parallel \ell_2$	4. Given two lines intercepted by a transversal, if a pair of corresponding angles are congruent, then the two lines are parallel.

• PROBLEM 3-7

If line $\overleftrightarrow{AB}$ is parallel to line $\overleftrightarrow{CD}$ and line $\overleftrightarrow{EF}$ is parallel to line $\overleftrightarrow{GH}$, prove that $m\measuredangle 1 = m\measuredangle 2$.

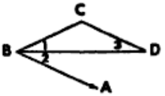


Solution: To show $\measuredangle 1 \cong \measuredangle 2$, we relate both to $\measuredangle 3$. Because $\overline{EF} \parallel \overline{GH}$, corresponding angles 1 and 3 are congruent. Since $\overline{AB} \parallel \overline{CD}$, corresponding angles 3 and 2 are congruent. Because both $\measuredangle 1$ and $\measuredangle 2$ are congruent to the same angle, it follows that $\measuredangle 1 \cong \measuredangle 2$.

Statements	Reasons
1. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$	1. Given
2. $m\measuredangle 1 = m\measuredangle 3$	2. If two parallel lines are cut by a transversal, corresponding angles are of equal measure.
3. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	3. Given
4. $m\measuredangle 2 = m\measuredangle 3$	4. If two parallel lines are cut by a transversal, corresponding angles are equal in measure.
5. $m\measuredangle 1 = m\measuredangle 2$	5. If two quantities are equal to the same quantity, they are equal to each other.

• PROBLEM 3-8

As shown in the accompanying figure, if $\overline{BD}$ bisects $\measuredangle ABC$ and $\overline{BC} \cong \overline{CD}$, prove formally that $\overleftrightarrow{CD} \parallel \overrightarrow{BA}$.



Solution: If two lines are cut by a transversal which form a pair of alternate interior angles that are congruent, then the two lines are parallel. The transversal of lines $\overleftrightarrow{CD}$ and $\overrightarrow{BA}$ is $\overleftrightarrow{BD}$, and $\measuredangle 2$ and $\measuredangle 3$ are alternate interior angles of this transversal. Our task is to prove $\measuredangle 2 \cong \measuredangle 3$. From the above theorem, it then follows that $\overrightarrow{BA} \parallel \overleftrightarrow{CD}$.

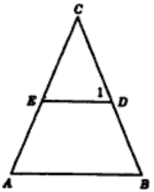
Statement	Reason
1. $\overline{BC} \cong \overline{CD}$	1. Given
2. $\measuredangle 3 \cong \measuredangle 1$	2. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
3. $\measuredangle 1 \cong \measuredangle 2$	3. An angle bisector divides the angle into two congruent angles.
4. $\measuredangle 3 \cong \measuredangle 2$	4. Transitive property of congruence.
5. $\overleftrightarrow{CD} \parallel \overrightarrow{BA}$	5. If two lines are cut by a transversal which forms a pair of congruent alternate interior angles, then the two lines are parallel.

• PROBLEM 3-9

Given: Δ ABC is isosceles with base $\overline{AB}$.

$\measuredangle A \cong \measuredangle 1$

Prove: AB $\parallel$ ED .

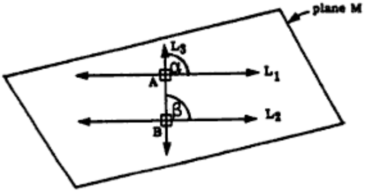


Solution: To show two lines parallel, it is sufficient to show that a pair of corresponding angles are congruent. Here, we use the fact that ΔACB is isosceles to show $\measuredangle A \cong \measuredangle B$. Combining this with the given, $\measuredangle 1 \cong \measuredangle A$, we obtain $\measuredangle B \cong \measuredangle 1$. Since these are corresponding angles of $\overline{ED}$ and $\overline{AB}$, it follows that $\overline{ED} \parallel \overline{AB}$.

Statements	Reasons
1. ΔABC is isosceles with base $\overline{AB}$	1. Given
2. $\overline{AC} \cong \overline{BC}$	2. Definition of an isosceles triangle.
3. $\measuredangle A \cong \measuredangle B$	3. If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
4. $\measuredangle A \cong \measuredangle 1$	4. Given
5. $\measuredangle B \cong \measuredangle 1$	5. If two quantities are congruent to the same quantity, they are congruent to each other.
6. $\overline{AB} \parallel \overline{ED}$	6. If two lines are cut by a transversal so that a pair of corresponding angles are congruent, the lines are parallel.

• PROBLEM 3-10

Let L_1, L_2, and L_3 be lines in plane M. Prove that if $L_1 \parallel L_2$ and $L_3 \perp L_2$, then $L_3 \perp L_1$.



Solution: There are 2 cases to consider:

I) L_3 intersects L_1;

II) L_3 does not intersect L_1.

Each case is treated separately.

Case I: The figure shows the situation for this case. We shall use the fact that the consecutive interior angles of 2 parallel lines cut by a transversal are supplementary to prove that $m \measuredangle \alpha = 90^\circ$. (i.e., $L_3 \perp L_1$).

First note that L_3 acts as a transversal for parallel lines L_1 and L_2. Angles α and BAC are consecutive interior angles for L_1 and L_2, and are therefore supplementary. Hence, their measures must sum to 180°, and we may write:

$$m \measuredangle BAC + m \measuredangle \alpha = 180^\circ \qquad (1)$$

But, as the problem indicates, $\measuredangle$ BAC is a right angle. Therefore,

$$m \measuredangle BAC = 90^\circ \;. \qquad (2)$$

Substituting (2) in (1) yields

$$m \measuredangle \alpha + 90^\circ = 180^\circ$$

or

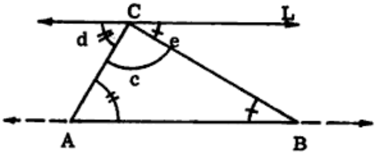
$$m \measuredangle \alpha = 90^\circ \;.$$

This implies that $L_3 \perp L_1$, as was to be shown.

Case II: In this situation, L_3 doesn't intersect L_1. But, if 2 lines don't intersect, they are parallel, i.e., $L_3 \parallel L_1$. However, we know (from the statement of the problem), that $L_1 \parallel L_2$. Coupling these 2 facts leads to the conclusion that $L_2 \parallel L_3$, which is contrary to the facts as given by the statement of the problem. Hence, this case need not be considered.

• PROBLEM 3-11

Prove the angle sum theorem: the sum of the measures of the angles of a triangle is 180°.



Solution: In the figure, L is constructed parallel to $\overline{AB}$ and it passes through C. Since $\overline{LC} \parallel \overline{AB}$, $\measuredangle d \cong \measuredangle CAB$ because they are alternate interior angles. Similarly, $\measuredangle e \cong \measuredangle CBA$.

The sum of the angles of the triangle is

$$S = m \measuredangle CAB + m \measuredangle ACB + m \measuredangle CBA \;.$$

Using the results found above,

$$S = m \measuredangle d + m \measuredangle ACB + m \measuredangle e. \qquad (1)$$

But

$$m\measuredangle d + m\measuredangle e + m\measuredangle c = 180^\circ$$

since they sum to a straight angle (segment $\overline{CL}$). Hence,

$$m\measuredangle d + m\measuredangle e = 180^\circ - m\measuredangle c \,. \qquad (2)$$

Substituting (2) into (1),

$$S = 180^\circ - m\measuredangle c + m\measuredangle ACB.$$

But, $\measuredangle ACB = \measuredangle c$. Thus,

$$S = 180^\circ.$$

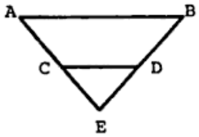
• PROBLEM 3-12

Given: $\triangle ABE$

$\overline{AB} \parallel \overline{CD}$

$\overline{CE} \cong \overline{DE}$

Prove: $AE = BE$.



Solution: If two angles of a triangle are congruent, then the two sides opposite them are congruent. To show $AE = BE$ in $\triangle ABE$, we show $\measuredangle A \cong \measuredangle B$. We prove this by the following steps: (1) Since $CE = ED$, $\measuredangle DCE \cong \measuredangle CDE$; (2) Since $\overline{AB} \parallel \overline{CD}$, the corresponding angles are congruent. Thus, $\measuredangle A \cong \measuredangle ECD$ and $\measuredangle B \cong \measuredangle CDE$; (3) Since $\measuredangle A$ and $\measuredangle B$ are congruent to congruent angles, $\measuredangle A \cong \measuredangle B$.

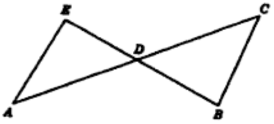
Statements	Reasons
1. $\overline{CE} \cong \overline{DE}$	1. Given
2. $m\measuredangle ECD = m\measuredangle CDE$	2. If two sides of a triangle have equal length, then the angles opposite those sides are of equal measure.
3. $\overline{AB} \parallel \overline{CD}$	3. Given
4. $m\measuredangle ECD = m\measuredangle A$	4. If two parallel lines are cut by a transversal, the corresponding angles are of equal measure.
5. $m\measuredangle CDE = m\measuredangle A$	5. If two quantities are equal to the same quantity, they are equal to each other.

6. $m\angle CDE = m\angle B$	6. If two parallel lines are cut by a transversal, the corresponding angles are of equal measure.
7. $m\angle A = m\angle B$	7. If two quantities are equal to the same quantity, they are equal to each other.
8. $AE = BE$	8. If a triangle has two angles of equal measure, then the sides opposite those angles are of equal lengths.

• PROBLEM 3-13

Given: $\overline{AC}$ and $\overline{EB}$ bisect each other at D.

Prove: $\overline{AE} \parallel \overline{BC}$.



Solution: To show two lines are parallel, it is sufficient to show that a pair of alternate interior angles, such as $\angle A$ and $\angle C$, are congruent.

We first prove $\Delta AED \cong \Delta CBD$ by the SAS postulate. Because corresponding parts of congruent triangles are congruent, $\angle A \cong \angle C$, and thus the lines are parallel.

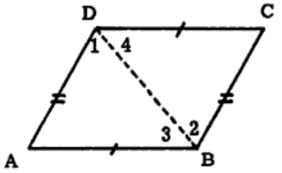
Statements	Reasons
1. $\overline{AC}$ and $\overline{EB}$ bisect each other at D.	1. Given
2. $\overline{ED} \cong \overline{BD}$ $\overline{AD} \cong \overline{CD}$	2. Definition of bisector.
3. $\angle EDA \cong \angle BDC$	3. Vertical angles are congruent.
4. $\Delta EDA \cong \Delta BDC$	4. SAS $\cong$ SAS
5. $\angle A \cong \angle C$	5. Corresponding parts of congruent triangles are congruent.
6. $\overline{AE} \parallel \overline{BC}$	6. If two lines are cut by a transversal so that alternate interior angles are congruent, the lines are parallel.

• PROBLEM 3-14

Prove that if both pairs of opposite sides of a quadrilateral are congruent, then they are also parallel.

Given: Quadrilateral ABCD; $\overline{AB} \cong \overline{CD}$; $\overline{AD} \cong \overline{BC}$

Prove: $\overline{AD} \parallel \overline{BC}$; $\overline{AB} \parallel \overline{CD}$



Solution: In the accompanying figure, the opposite sides of quadrilateral ABCD are congruent. Thus, $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. We must show $\overline{DC} \parallel \overline{AB}$ and $\overline{AD} \parallel \overline{BC}$.

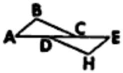
To do this, we draw diagonal $\overline{DB}$. Remember that if alternate interior angles are congruent, the two lines are parallel. Thus, to show AB $\parallel$ CD, we prove $\angle 3 \cong \angle 4$. To show $\angle 3 \cong \angle 4$, we prove Δ ADB $\cong \Delta$ CBD by the SSS Postulate. Thus, by corresponding parts, $\angle 3 \cong \angle 4$ and $\overline{AB} \parallel \overline{DC}$.

By corresponding parts, we can also say $\angle 1 \cong \angle 2$. Since $\angle 1$ and $\angle 2$ we alternate interior angles of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$, it follows that $\overline{AD} \parallel \overline{BC}$.

Statements	Reasons
1. Quadrilateral ABCD; $\overline{AB} \cong \overline{CD}$; $\overline{AD} \cong \overline{CD}$	1. Given.
2. $\overline{DB} \cong \overline{DB}$	2. A segment is congruent to itself.
3. Δ ADB $\cong \Delta$ CBD	3. The SSS Postulate.
4. $\angle 1 \cong \angle 2$ $\angle 3 \cong \angle 4$	4. Corresponding angles of congruent triangles are congruent.
5. $\overline{AD} \parallel \overline{BC}$, $\overline{AB} \parallel \overline{CD}$	5. If two coplanar lines are cut by a transversal such that the alternate interior angles are congruent, then the lines are paralel.

• PROBLEM 3-15

We are given two triangles, ABC and EHD, as shown in the diagram, and the straight line segment $\overline{AE}$. If $\overline{AD} \cong \overline{EC}$, $\overline{BC} \cong \overline{HD}$ and $\overline{BC} \parallel \overline{HD}$, prove $\overline{AB} \parallel \overline{EH}$.



Solution: To prove $\overline{AB} \parallel \overline{EH}$, we first show that the alternate interior angles, $\angle$ BAC and $\angle$ HED, formed by transversal $\overline{AE}$ and segments $\overline{AB}$ and $\overline{EH}$, are congruent. Since a pair of congruent alternate interior angles are sufficient to prove two lines parallel, we may conclude that the $\overline{AB} \parallel \overline{EH}$.

To prove $\angle$ BAC and $\angle$ HED are congruent, we must first prove the triangles containing them, Δ's ABC and EHD, are congruent by the SAS $\cong$ SAS method.

Statement	Reason
1. $\overline{BC} \cong \overline{HD}$	1. Given
2. $\overline{BC} \parallel \overline{HD}$	2. Given
3. $\angle$ BCA $\cong$ $\angle$ HDE	3. If two parallel lines are cut by a transversal, then alternate interior angles are congruent.
4. $\overline{AE}$ is a line segment.	4. Given
5. $\overline{AD} \cong \overline{EC}$	5. Given
6. $\overline{DC} \cong \overline{DC}$	6. Reflexive property of congruence.
7. AD = EC; DC = DC	7. Congruent segments are of equal lengths.
8. AD + DC = EC + CD	8. Addition Postulate
9. AC = ED	9. Point Betweenness Postulate.
10. $\overline{AC} \cong \overline{ED}$	10. Segments of equal length are congruent.
11. ΔABC $\cong$ ΔEHD	11. SAS $\cong$ SAS
12. $\angle$ BAC $\cong$ $\angle$ HED	12. Corresponding parts of congruent triangles are congruent.
13. $\overline{AB} \parallel \overline{EH}$	13. If two lines are cut by a transversal making a pair of alternate interior angles congruent, then the lines are parallel.

• PROBLEM 3-16

In Figure 1, $\angle$ B measures 30°, $\overleftrightarrow{BQ} \parallel \overleftrightarrow{AP}$ and $\overleftrightarrow{BP} \parallel \overleftrightarrow{AC}$. Determine m $\angle$ 1, m $\angle$ 2, m $\angle$ 3, m $\angle$ 4, and m $\angle$ 5.

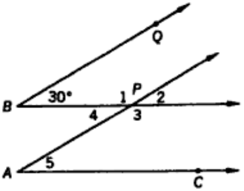


Figure 1

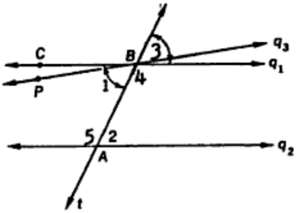


Figure 2

Solution: Prior to attacking the question at hand, we will establish several important facts about angles formed by parallel lines cut by a transversal. We will employ two characteristics about parallel lines: (1) Through a given external point, only one line can be drawn parallel to a given line; and (2) If one pair of alternate interior angles are congruent, then the two lines must be parallel. (See Figure 2).

The first fact we wish to show is that, given two parallel lines q_1 and q_2 cut by transversal t, every pair of alternate interior angles are congruent. We do this by indirect proof. Suppose $q_1 \| q_2$ but alternate interior angles $\measuredangle 1$ and $\measuredangle 2$ are not congruent. Draw q_3 through point B such that it forms $\measuredangle ABP$ so that $\measuredangle ABP \cong \measuredangle 2$. This leads to two contradictory lines of reasoning:

(A) Because $\measuredangle ABP \cong \measuredangle 2$ and $\measuredangle 1$ is not $\cong \measuredangle 2$, it follows that $\measuredangle ABP$ is not $\cong \measuredangle 1$. Since q_3 and q_1 make different angles with transversal t, lines q_1 and q_3 must be different lines.

(B) Because alternate interior angles $\measuredangle ABP$ and $\measuredangle 2$ are congruent, (by assumption (2)) it follows that $q_3 \| q_2$. Since both q_3 and q_1 are lines parallel to q_2 and passing through external point B, (by assumption (1)) it follows that q_1 and q_3 are the same line.

Because the two lines of reasoning lead to different conclusions, our original assumption - $\measuredangle 1$ is not $\cong \measuredangle 2$ - must be false. We can thus conclude that if two parallel lines are cut by a transversal, any pair of alternate interior angles are congruent.

Next we shall establish that if two parallel lines are cut by a transversal, each pair of corresponding angles are congruent. From the above theorem, we know $\measuredangle 1 \cong \measuredangle 2$. Since $\measuredangle 1$ and $\measuredangle 3$ are, by definition, vertical angles, then $\measuredangle 1 \cong \measuredangle 3$. Therefore, by transitivity, $\measuredangle 2 \cong \measuredangle 3$ and we have the desired results.

Finally, we will prove that interior angles on the same side of the transversal are supplementary.

Again, since alternate interior angles of parallel lines cut by a transversal are congruent, we have $\measuredangle 1 \cong \measuredangle 2$. Adjacent angles, whose exterior sides are contained in a straight line, are supplementary. Thus, their measures sum to 180°. Therefore,

$$m\measuredangle 1 + m\measuredangle 4 = 180$$

$$m\measuredangle 2 + m\measuredangle 5 = 180 .$$

Since $m\measuredangle 1 = m\measuredangle 2$, the quantities can be interchanged to obtain the following:

$$m\measuredangle 2 + m\measuredangle 4 = 180$$

$$m\measuredangle 1 + m\measuredangle 5 = 180 .$$

Angles 2 and 4, as well as ∡ 1 and ∡ 5, are interior angles on the same side of the transversal, and since their measures sum to 180° , they must be supplementary to each other. Therefore, we have established our third and final point.

Now we return to the original question. (See Figure 1.)

Since $\overleftrightarrow{BQ} \parallel \overleftrightarrow{AP}$ and ∡ B and ∡ 1 are interior angles on the same side of the transversal, by a theorem above, ∡ B and ∡ 1 are supplements, i.e. m ∡ B + m ∡ 1 = 180 . We are given m ∡ B = 30. Therefore, by substitution, 30 + m ∡ 1 = 180, and m ∡ 1 = 150.

Angles B and 2 are corresponding angles formed by a transversal cutting parallel lines $\overleftrightarrow{BQ}$ and $\overleftrightarrow{AP}$. As such, m ∡ B = m ∡ 2. Therefore, m ∡ 2 = 30.

For the parallel lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{AC}$ cut by transversal $\overleftrightarrow{AP}$, ∡ 5 and ∡ 2 are corresponding angles and, accordingly, are congruent. Therefore, m ∡ 5 = 30.

On the same set of parallel lines, ∡ 5 and ∡ 4 are alternate interior angles and, thus are congruent. Therefore, m ∡ 4 = 30.

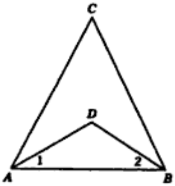
The last angle we need to determine is ∡ 3. Angle 3 is on the same side of the transversal as ∡ 5 and on the interior of $\overleftrightarrow{BP} \parallel \overleftrightarrow{AC}$. Therefore, ∡ 3 is supplementary to ∡ 5. As such, m ∡ 3 + m ∡ 5 = 180. By substitution, m ∡ 3 + 30 = 180. Thus, m ∡ 3 = 150.

Collecting our results, we have: m ∡ 1 = 150, m ∡ 2 = 30, m ∡ 3 = 150, m ∡ 4 = 30, and m ∡ 5 = 30.

ANGLE BISECTORS

• PROBLEM 3-17

Given: DA bisects ∡ CAB. DB bisects ∡ CBA.
m ∡ 1 = m ∡ 2 .
Prove: CA = CB.

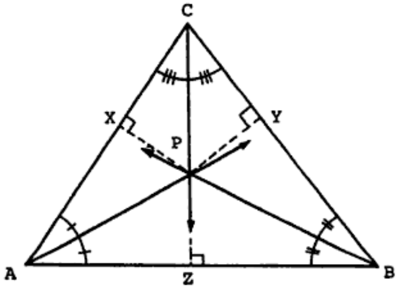


Solution: This proof will involve using the definition of angle bisector and the substitution and addition postulates to prove m ∡ CAB = m ∡ CBA. The desired results will be obtained recalling that the sides of a triangle opposite angles of equal measure are of equal length.

Statements	Reasons
1. DA bisects ∢ CAB DB bisects ∢ CBA	1. Given
2. m ∢ CAD = m ∢ 1 m ∢ CBD = m ∢ 2	2. An angle bisector divides an angle into two angles of equal measure.
3. m ∢ = 1 = m ∢ 2	3. Given
4. m ∢ CBD = m ∢ 1	4. A quantity may be substituted for its equal.
5. m ∢ CAD = m ∢ CBD	5. If two quantities are equal to the same quantity, or equal quantities, then they are equal to each other.
6. m ∢ CAD + m ∢ 1 = m ∢ CBD + m ∢ 2	6. If equal quantities are added to equal quantities, the sums are equal quantities.
7. m ∢ CAB = m ∢ CAD + m ∢ 1 m∢CBA = m∢CBD + m∢2	7. The measure of the whole is equal to the sum of the measures of its parts.
8. m ∢ CAB = m ∢ CBA	8. If two quantities are equal to the same quantity, or equal quantities, then they're equal to each other.
9. CA = CB	9. If a triangle has two angles of equal measure then the sides opposite those angles are equal in length.

• PROBLEM 3-18

Show that the angle bisectors of a triangle are concurrent at a point equidistant from the sides of the triangle.



Solution: The angle bisector of any angle is the set of points in the interior of the angle that are equidistant from the sides of the angle.

In ΔABC, the intersection of the bisector of ∢ A and the bisector of ∢ B is a point P. Because P is on the bisector of ∢ A, the distance from P to $\overline{AC}$ equals the distance from P to $\overline{AB}$. Because P is on the bisector of ∢ B, the distance from P to $\overline{BC}$ equals the distance from P to $\overline{AB}$. By transitivity, the distance from P to $\overline{AC}$ equals the distance from P to $\overline{BC}$, i.e. P is equidistant from the sides of ∡ C. Therefore, point P, is also on the bisector of ∡ C, and the three bisectors are concurrent at a point equidistant from the sides of the triangle.

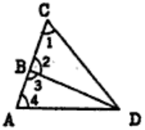
Given: The angle bisectors of △ABC .
Prove: The bisectors are concurrent at a point equidistant from the sides.

Statements	Reasons
1. The angle bisectors of △ABC	1. Given
2. Let P be the point of intersection of the bisectors of ∡A and ∡B.	2. In a plane, two non-parallel, non-coincident lines intersect in a unique point.
3. P is in the interior of ∡ A. P is in the interior of ∡ B.	3. All points (except for the vertex) of the angle bisector lie in the interior of the angle.
4. P is in the interior of △ABC	4. If a point is in the interior of two angles of a triangle, then it is in the interior of the triangle.
5. Let X,Y, and Z be the points on sides $\overline{AC}$, $\overline{CB}$, and $\overline{AB}$ such that $\overline{PX} \perp \overline{AC}$, $\overline{PY} \perp \overline{CB}$, and $\overline{PZ} \perp \overline{AB}$.	5. From a given external point, only one line can be drawn perpendicular to a given line.
6. PX, PY, and PZ are the distances from P to the sides	6. The distance from an external point to a line is the length of the perpendicular segment from the point to the line.
7. PX = PZ PY = PZ	7. All points on the angle bisector are equidistant from the sides of the angle.
8. PX = PY	8. Transitivity Postulate.
9. P is in the interior of ∡ C.	9. All points in the interior of a triangle are in the interior of each of the angles.
10. P is on the angle bisector of ∡ C.	10. All points in the interior of an angle that are equidistant from the sides of the angle are on the angle bisector.
11. P is equidistant from the sides of △ABC.	11. Follows from Steps 7 and 8.
12. The angle bisectors are concurrent at a point equidistant from the sides.	12. Lines are concurrent if their intersection consists of at least one point. Also, Step 11.

EXTERIOR ANGLES

• PROBLEM 3-19

In the figure, what can be said about ∡3 and ∡1? About ∡2 and ∡4? If m∡4 > m∡3, prove that m∡2 > m∡1.(See figure.)



Solution: We use the Exterior Angle Theorem to find relationships between the given angles. This theorem states that an exterior angle of a triangle has greater measure than either of its remote interior angles. (An exterior angle of a triangle is any angle outside of the triangle which forms a linear pair with an angle of the triangle. A remote interior angle of a triangle is an angle of a triangle which does not form a linear pair with the given exterior angle in question.)

In the figure, $\angle 3$ is an exterior angle of ΔBCD. $\angle 1$ is a remote interior angle of the triangle, relative to angle 3. Hence, $m\angle 3 > m\angle 1$. Furthermore, $\angle 2$ is an exterior angle of ΔBAD, and $\angle 4$ is a remote interior angle of ΔBAD relative to $\angle 2$. Hence, $m\angle 2 > m\angle 4$.

Given that $m\angle 4 > m\angle 3$, and using the conclusions drawn above, we may write the following string of inequalities:

$$\begin{aligned} m\angle 2 &> m\angle 4 \\ m\angle 4 &> m\angle 3 \\ m\angle 3 &> m\angle 1 \end{aligned} \qquad (1)$$

Coupling the first two of these inequalities yields

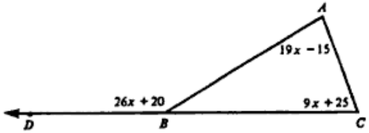
$m\angle 2 > m\angle 3$ and $m\angle 3 > m\angle 1$.

Combining these 2 inequalities gives

$m\angle 2 > m\angle 1$.

• PROBLEM 3-20

Find the numerical measure of each of the angles of ΔABC if $m\angle A = 19x - 15$, $m\angle C = 9x + 25$, and $m\angle ABD = 26x + 20$.



Solution: To find $m\angle A$, $m\angle B$ and $m\angle C$, we first solve for › x can be solved for by using the Exterior Angle Theorem: the measure of an exterior angle of the triangle equals the sum of the remote interior angles. Therefore,

$$m\angle ABD = m\angle A + m\angle C$$

$$26x + 20 = (19x - 15) + (9x + 25) = 28x + 10$$

To isolate the variable x, we subtract 26x and 10 from both sides.

$$2x = 10 \qquad \text{or} \qquad x = 5.$$

Then $m\angle A = 19x - 15 = 19(5) - 15 = 80°$

$m\angle C = 9x + 25 = 9(5) + 25 = 70°$

Since, $\angle A$, $\angle B$, and $\angle C$ are angles of the same triangle,

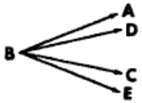
$$m\angle A + m\angle B + m\angle C = 180°$$

or, $m\angle B = 180 - m\angle A - m\angle C = 180 - 80 - 70 = 30°$.

CONGRUENT ANGLES

• PROBLEM 3-21

$\angle ABE$ is intersected by rays BD and BC in such a way that $\angle ABC \cong \angle DBE$. Prove that $\angle ABD \cong \angle CBE$.



Solution: This proof will first employ the reflexive property of angles (An angle is congruent to itself) and then the Subtraction Postulate (Equal quantities subtracted from equal quantities yield equal quantities).

GIVEN: $\angle ABC \cong \angle DBE$

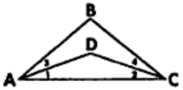
PROVE: $\angle ABD \cong \angle CBE$

STATEMENTS	REASONS
1. $\angle ABC \cong \angle DBE$	1. Given
2. $\angle DBC \cong \angle DBC$	2. An angle is congruent to itself.
3. $m\angle ABC = m\angle DBE$ $m\angle DBC = m\angle DBC$	3. Congruent angles have equal measures.

4. $m\angle ABC - m\angle DBC = m\angle DBE - m\angle DBC$	4. Subtraction Postulate.
5. $m\angle ABD = m\angle CBE$	5. Substitution Posulate.
6. $\angle ABD \cong \angle CBE$	6. Angles of equal measure are congruent.

• PROBLEM 3-22

In the figure shown, $\triangle ABC$ is an isosceles triangle, such that $\overline{BA} \cong \overline{BC}$. Line segment $\overline{AD}$ bisects $\angle BAC$ and $\overline{CD}$ bisects $\angle BCA$. Prove that $\triangle ADC$ is an isosceles triangle.



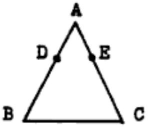
Solution: In order to prove $\triangle ADC$ is isosceles, we must prove that 2 of its sides, $\overline{AD}$ and $\overline{CD}$, are congruent. To prove $\overline{AD} \cong \overline{CD}$ in $\triangle ADC$, we have to prove that the angles opposite $\overline{AD}$ and $\overline{CD}$, $\angle 1$ and $\angle 2$, are congruent.

STATEMENT	REASON
1. $\overline{BA} \cong \overline{BC}$	1. Given.
2. $\angle BAC \cong \angle BCA$ or $m\angle BAC = m\angle BCA$	2. If two sides of a triangle are congruent, then the angles opposite them are congruent.
3. $\overline{AD}$ bisects $\angle BAC$ $\overline{CD}$ bisects $\angle BCA$	3. Given.
4. $m\angle 1 = \frac{1}{2} m\angle BAC$ $m\angle 2 = \frac{1}{2} m\angle BCA$	4. The bisector of an angle divides the angle into two angles whose measures are equal.
5. $m\angle 1 = m\angle 2$	5. Halves of equal quantities are equal.
6. $\angle 1 \cong \angle 2$	6. If the measure of two angles are equal, then the angles are congruent.
7. $\overline{CD} \cong \overline{AD}$	7. If two angles of a triangle are congruent, then the sides opposite these angles are congruent.
8 $\triangle ADC$ is an isosceles triangle	8. If a triangle has two congruent sides, then it is an isosceles triangle.

CONGRUENT LINE SEGMENTS

• PROBLEM 3-23

Given isosceles triangle ABC with sides $\overline{AB} \cong \overline{AC}$, and the fact that $\overline{DB} \cong \overline{EC}$, prove that $\overline{AD} \cong \overline{AE}$.

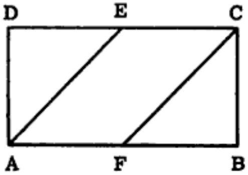


Solution: In this problem, we make use of the Subtraction Postulate as applied to congruency. The Subtraction Postulate states that when equal quantities are subtracted from equal quantities, the differences are equal.

STATEMENT	REASON
1. $\overline{AB} \cong \overline{AC}$ $\overline{DB} \cong \overline{EC}$	1. Given.
2. $AB = AC$ $DB = EC$	2. Congruent segments are equal in length.
3. $AB - DB = AC - EC$	3. Subtraction Postulate.
4. $AD = AE$	4. Substitution Postulate.
5. $\overline{AD} \cong \overline{AE}$	5. Segments of equal lengths are congruent.

• PROBLEM 3-24

Lines have been drawn in rectangle ABCD from point A bisecting $\overline{CD}$, and from point C, bisecting $\overline{AB}$ (as in the figure). In a rectangle, opposite sides are of equal length. Therefore, $\overline{AB} \cong \overline{DC}$. Prove $\overline{AF} \cong \overline{EC}$.



Solution: The Division Postulate states that congruent quantities divided by equal nonzero quantities result in quotients that are equal. This postulate will be employed in the following proof.

GIVEN: $\overline{AB} \cong \overline{DC}$; $\overline{AE}$ bisects $\overline{DC}$; $\overline{CF}$ bisects $\overline{AB}$

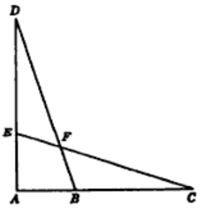
PROVE: $\overline{AF} \cong \overline{EC}$

STATEMENT	REASON
1. $\overline{AB} \cong \overline{DC}$; $\overline{AE}$ bisects $\overline{DC}$; $\overline{CF}$ bisects $\overline{AB}$	1. Given.
2. $EC = \frac{1}{2}\,DC$ $AF = \frac{1}{2}\,AB$	2. The bisector of a segment divides it into two congruent segments, each of whose measure is one half the measure of the original segment.
3. $EC = AF$	3. The Division Postulate.
4. $\overline{AF} \cong \overline{EC}$	4. Line segments of equal length are congruent.

SIDE - ANGLE - SIDE POSTULATE

• PROBLEM 3-25

Given: $\overline{AD} \cong \overline{AC}$. $\overline{AB} \cong \overline{AE}$. Prove: $\triangle ADB \cong \triangle ACE$.

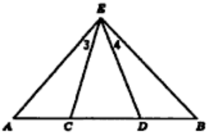


Solution: The fact that the relevant triangles overlap will aid us in completing this proof. We are given congruence between two pairs of corresponding sides. Furthermore the angle included between the sides is shared by the two triangles. By the SAS Postulate, the two triangles are congruent.

STATEMENTS	REASONS
1. $\overline{AD} \cong \overline{AC}$	1. Given.
2. $\overline{AB} \cong \overline{AE}$	2. Given.
3. $\measuredangle A \cong \measuredangle A$	3. Reflexive Property of Congruence; a quantity is congruent to itself.
4. $\Delta ADB \cong \Delta ACE$	4. SAS $\cong$ SAS; if two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

• PROBLEM 3-26

Given: ΔABE, $\overline{AE} \cong \overline{BE}$, $\overline{AC} \cong \overline{BD}$. Prove: $\measuredangle 3 \cong \measuredangle 4$.



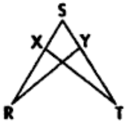
Solution: Angles 3 and 4 are corresponding parts of Δ's EAC and EBD. We can prove the required congruence, therefore, by showing $\Delta EAC \cong \Delta EBD$.

From the given information, we have congruence between two pairs of corresponding sides. We can use the fact that the angles opposite the congruent sides of a triangle are congruent to show congruence of the corresponding included angles. Hence, by the SAS posulate, we will have proved $\Delta EAC \cong \Delta EBD$.

STATEMENTS	REASONS
1. $\overline{AE} \cong \overline{BE}$	1. Given.
2. $\measuredangle A \cong \measuredangle B$	2. If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
3. $\overline{AC} \cong \overline{BD}$	3. Given.
4. $\Delta EAC \cong \Delta EBD$	4. SAS $\cong$ SAS
5. $\measuredangle 3 \cong \measuredangle 4$	5. Corresponding parts of congruent triangles are congruent.

• PROBLEM 3-27

In the figure, $\overline{SR}$ and $\overline{ST}$ are straight line segments. $\overline{SX} \cong \overline{SY}$ and $\overline{XR} \cong \overline{YT}$. Prove that $\Delta RSY \cong \Delta TSX$.



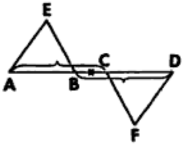
Solution: The triangles are overlapping in this problem because they share a common angle.

This proof will employ the S.A.S. Postulate for showing congruence between two triangles.

STATEMENT	REASON
1. $\overline{SX} \cong \overline{SY}$	1. Given.
2. $\overline{XR} \cong \overline{YT}$	2. Given.
3. $SX = SY$; $XR = YT$	3. Congruent segments are of equal length.
4. $SX + XR = SY + YT$ or $SR = ST$	4. Addition Postulate
5. $\overline{SR} \cong \overline{ST}$	5. Segments of equal length are congruent.
6. $\measuredangle S \cong \measuredangle S$	6. Reflexive Property of Congruence.
7. $\Delta RSY \cong \Delta TSX$	7. S.A.S. $\cong$ S.A.S.

• PROBLEM 3-28

$\overline{AD}$ is a straight line segment. Triangles AEB and DFC are drawn in such a way that $\measuredangle A \cong \measuredangle D$ and $\overline{AE} \cong \overline{DF}$. Additionally, it is given that $\overline{AC} \cong \overline{BD}$. Prove that $\Delta AEB \cong \Delta DFC$.



Solution: Since we are given some information about two sides and the included angle, it seems logical to use the postulate which states that if two sides of a triangle and the included angle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

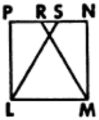
GIVEN: $\angle A \cong \angle D$; $\overline{AE} \cong \overline{DF}$; $\overline{AC} \cong \overline{BD}$

Prove: $\Delta AEB \cong \Delta DFC$

STATEMENT	REASON
1. $\overline{AE} \cong \overline{DF}$	1. Given.
2. $\angle A \cong \angle D$	2. Given.
3. $\overline{AD}$ is a straight line segment	3. Given.
4. $\overline{AC} \cong \overline{DB}$	4. Given.
5. $\overline{BC} \cong \overline{BC}$	5. Reflexive property of congruence
6. $\overline{AC} - \overline{BC} \cong \overline{DB} - \overline{BC}$ or $\overline{AB} \cong \overline{DC}$	6. If congruent segments are subtracted from congruent segments the differences are congruent segments.
7. $\Delta AEB \cong \Delta DFC$	7. S.A.S. $\cong$ S.A.S.

• PROBLEM 3-29

We are given quadrilateral LMNP with $\overline{LS}$ and $\overline{MR}$ drawn as shown in the figure. If $\overline{LP} \perp \overline{PN}$, $\overline{MN} \perp \overline{PN}$, $\overline{LP} \cong \overline{MN}$ and $\overline{PR} \cong \overline{NS}$, then prove that $\Delta LPS \cong \Delta MNR$.



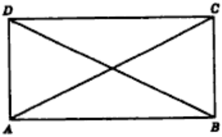
Solution: Triangles LPS and MNR overlap in the sense that $\overline{RS}$ is shared by both side PS of ΔLPS and side RN of ΔMNR. This feature will aid us in proving the congruence required by the problem.

STATEMENT	REASON
1. $\overline{LP} \cong \overline{MN}$	1. Given.
2. $\overline{LP} \perp \overline{PN}$ and $\overline{MN} \perp \overline{PN}$	2. Given.

3. $\angle LPS$ and $\angle MNR$ are right angles	3. Perpendicular lines intersect and form right angles.
4. $\angle LPS \cong \angle MNR$	4. All right angles are congruent.
5. $\overline{RS} \cong \overline{SR}$	5. Reflexive Property of Congruence.
6. $\overline{PR} \cong \overline{NS}$	6. Given.
7. $RS = SR$; $PR = NS$	7. Congruent segments are of equal length.
8. $RS + PR = SR + NS$ or $SP = NR$	8. Addition Postulate.
9. $\overline{SP} \cong \overline{NR}$	9. Segments of equal length are congruent.
10. $\triangle LPS \cong \triangle MNR$	10. SAS Postulate.

• PROBLEM 3-30

Given: $\overline{DC} \cong \overline{AB}$. $\angle CDA \cong \angle BAD$. Prove: $\overline{AC} \cong \overline{BD}$.

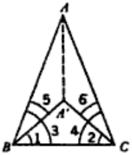


Solution: Noticing that $\overline{AC}$ and $\overline{BD}$ are corresponding parts of overlapping triangles CDA and BAD will assist us in completing this proof. We are given congruence between two corresponding parts and will use the fact that the triangles overlap to gain the third congruence.

STATEMENTS	REASONS
1. $\overline{DC} \cong \overline{AB}$	1. Given.
2. $\angle CDA \cong \angle BAD$	2. Given.
3. $\overline{DA} \cong \overline{DA}$	3. Reflexive Property of Congruence.
4. $\triangle CDA \cong \triangle BAD$	4. SAS $\cong$ SAS.
5. $\overline{AC} \cong \overline{BD}$	5. Corresponding parts of congruent triangles are congruent.

• PROBLEM 3-31

In the figure, $\overline{AB} \cong \overline{AC}$ and $\overline{A'B} \cong \overline{A'C}$. Using the SAS Postulate prove that $\Delta AA'B \cong \Delta AA'C$.



Solution: We shall use the fact that the base angles of an isosceles triangle are congruent to prove that $\angle 5 \cong \angle 6$ (see figure). Using this and the given facts we can then prove that $\Delta AA'B \cong \Delta AA'C$ by the S.A.S. Postulate.

Because $\overline{AB} \cong \overline{AC}$ and $\overline{A'B} \cong \overline{A'C}$, ΔABC and $\Delta A'BC$ are both isosceles. Since the base angles of an isosceles triangle are congruent, we obtain (see figure)

$$\angle 1 \cong \angle 2 \qquad (1)$$

$$\angle 3 \cong \angle 4 \qquad (2)$$

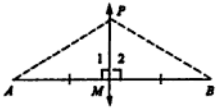
Subtracting (1) from (2) yields

$$(\angle 3 - \angle 1) \cong (\angle 4 - \angle 2) \quad \text{or} \quad \angle 5 \cong \angle 6.$$

From the given facts, $\overline{AB} \cong \overline{AC}$ and $\overline{A'B} \cong \overline{A'C}$. Hence, by the S.A.S. Postulate, $\Delta AA'B \cong \Delta AA'C$.

• PROBLEM 3-32

Prove that any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the segment.



Solution: Let A and B be the endpoints and M the midpoint of the line segment. Let P be any point on the bisector. The distances from P to the endpoints are AP and BP.

AP and BP are corresponding sides of triangles ΔAPM and ΔBPM. We show $\Delta APM \cong \Delta BPM$ by the SAS Postulate and thus AP must equal BP. Since P is an arbitrary point on the bisector, then all points of the perpendicular bisector must be equidistant from the endpoints of the bisected segment.

GIVEN: P is any point on the perpendicular bisector of $\overline{AB}$; the bisector and $\overline{AB}$ intersect at point M.

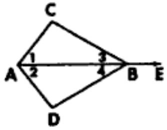
PROVE: PA = PB

STATEMENTS	REASONS
1. P is any point on the perpendicular bisector of $\overline{AB}$. The bisector and $\overline{AB}$ intersect at M	1. Given.
2. $\overleftrightarrow{PM} \perp \overline{AB}$	2. The perpendicular bisector of a segment is the line perpendicular to the segment at its midpoint.
3. $\sphericalangle 1$ and $\sphericalangle 2$ are right angles	3. Perpendicular lines intersect at right angles.
4. $\sphericalangle 1 \cong \sphericalangle 2$	4. All right angles are congruent.
5. M is the midpoint of $\overline{AB}$	5. Definition of perpendicular bisector.
6. AM = MB	6. The midpoint of a segment divides the segment into two equal segments.
7. Draw the auxiliary segments $\overline{PA}$ and $\overline{PB}$.	7. Any two distinct points determine a line (Line Postulate).
8. $\overline{PM} \cong \overline{PM}$	8. Any segment is congruent to itself.
9. $\Delta PAM \cong \Delta PBM$	9. The SAS Postulate.
10. PA = PB	10. Corresponding sides of congruent triangles are congruent.

ANGLE - SIDE - ANGLE POSTULATE

• PROBLEM 3-33

We are given, as shown in the figure, ΔACB and ΔADB and are told that straight line $\overleftrightarrow{AE}$ bisects $\angle CAD$. Also given is $\angle CBE \cong \angle DBE$. Prove $\Delta ACB \cong \Delta ADB$.



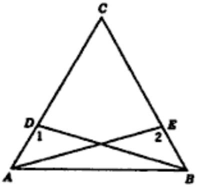
Solution: The most direct way to prove congruence in this case is to make use of the ASA Postulate. This postulate tells us that if two angles and the included side of one triangle are congruent, respectively, to two angles and the included side of another triangle, then the two triangles are congruent.

STATEMENT	REASON
1. $\overleftrightarrow{AE}$ bisects $\angle CAD$	1. Given.
2. $\angle 1 \cong \angle 2$	2. An angle bisector divides the angle into two congruent angles.
3. $\overline{AB} \cong \overline{AB}$	3. Reflexivity of congruence.
4. $\overleftrightarrow{AE}$ is a straight line	4. Given.
5. $\angle CBE \cong \angle DBE$	5. Given.
6. $\angle 3$ is supplementary to $\angle CBE$ and $\angle 4$ is supplementary to $\angle DBE$	6. If the non-common sides of two adjacent angles lie on a straight line, then the angles are supplementary.
7. $\angle 3 \cong \angle 4$	7. Supplements of congruent angles are congruent.
8. $\Delta ACB \cong \Delta ADB$	8. A.S.A. Postulate.

• PROBLEM 3-34

Given: ΔABC. $DC = EC$. $\angle 1 = \angle 2$. Prove: $\overline{AC} \cong \overline{BC}$.

Solution: The segments in question, $\overline{AC}$ and $\overline{BC}$, are corresponding parts of overlapping Δ's CDB and CEA. The proof is set up using the ASA postulate to show $\Delta CDB \cong \Delta CEA$.

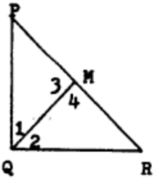


STATEMENTS	REASONS
1. $\angle 1 = \angle 2$	1. Given.
2. $\angle 1$ and $\angle CDB$ are supplements $\angle 2$ and $\angle CEA$ are supplements	2. Adjacent angles whose non-common sides form a straight line are supplements.
3. $\angle CDB = \angle CEA$	3. Supplements of equal angles are equal.
4. $\angle C = \angle C$	4. Reflexive law.
5. $\overline{DC} \cong \overline{EC}$	5. Given.
6. $\Delta CDB \cong \Delta CEA$	6. ASA $\cong$ ASA.
7. $\overline{AC} \cong \overline{BC}$	7. Corresponding parts of congruent triangles are congruent.

• PROBLEM 3-35

If $\angle 3 \cong \angle 4$ and $\overline{QM}$ bisects $\angle PQR$, prove that M is the midpoint of $\overline{PR}$.

Solution: To show that M is the midpoint of $\overline{PR}$, we prove that M divides $\overline{PR}$ into two congruent segments - that is, $\overline{PM} \cong \overline{MR}$. If we prove $\Delta PQM \cong \Delta RQM$, $\overline{PM} \cong \overline{MR}$ follows by corresponding parts.



We prove $\Delta PQM \cong \Delta RQM$ by the ASA Postulate: (1) $\angle 1 \cong \angle 2$ because $\overline{QM}$ bisects $\angle PQR$; (2) $\overline{QM} \cong \overline{QM}$; and (3) $\angle 3 \cong \angle 4$ is given. We present our results in the formal, two column proof format..

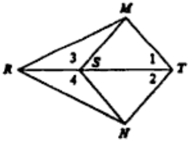
GIVEN: $\angle 3 \cong \angle 4$; $\overline{QM}$ bisects $\angle PQR$.

PROVE: M is the midpoint $\overline{PR}$.

STATEMENTS	REASONS
1. $\angle 3 \cong \angle 4$; $\overline{QM}$ bisects $\angle PQR$	1. Given.
2. $\angle 1 \cong \angle 2$	2. The angle bisector of an angle divides the angle into two congruent angles.
3. $\overline{QM} \cong \overline{QM}$	3. Every segment is congruent to itself.
4. $\Delta PQM \cong \Delta RQM$	4. ASA Postulate.
5. $\overline{PM} \cong \overline{RM}$	5. Corresponding sides of congruent triangles are congruent.
6. M is the midpoint of PR	6. Definition of a midpoint.

• PROBLEM 3-36

Given: $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$. Prove: RM = RN.



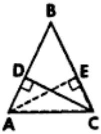
Solution: In most proofs, the necessary congruences are not always given. Frequently, to show two triangles congruent, we have to prove the congruence of corresponding parts by proving another set of triangles congruent. Here, we must show RM = RN or $\overline{RM} \cong \overline{RN}$. $\overline{RM}$ and $\overline{RN}$ are corresponding sides of ΔRMS and ΔRNS. In these triangles, we have that $\angle 3 \cong \angle 4$ and $\overline{RS} \cong \overline{RS}$. To prove congruence, we must also show $\overline{MS} \cong \overline{NS}$. We can show $\overline{MS} \cong \overline{NS}$ by proving $\Delta MST \cong \Delta NST$. Thus we (1) prove $\Delta MST \cong \Delta NST$ by the ASA Postulate; (2) show $\overline{MS} \cong \overline{NS}$ by corresponding parts; and (3) show $\Delta RMS \cong \Delta RNS$ by the SAS Postulate.

STATEMENTS	REASONS
1. $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	1. Given.
2. $\angle MST$ and $\angle 3$ are supplementary $\angle NST$ and $\angle 4$ are supplementary	2. Two angles that form a linear pair are supplementary.

3. ∠MST ≅ ∠NST	3. Angles supplementary to congruent angles are congruent.
4. $\overline{ST} \cong \overline{ST}$	4. A segment is congruent to itself.
5. △SMT ≅ △SNT	5. The ASA Postulate.
6. $\overline{MS} \cong \overline{NS}$	6. Corresponding sides of congruent triangles are congruent.
7. $\overline{RS} \cong \overline{RS}$	7. A segment is congruent to itself.
8. △RMS ≅ △RNS	8. The SAS Postulate.
9. $\overline{RM} \cong \overline{RN}$	9. Corresponding sides of congruent triangles are congruent.
10. RM = RN	10. Congruent segments are of the same length.

• PROBLEM 3-37

Prove that the altitudes drawn to the legs of an isosceles triangle are congruent.



Solution: The accompanying figure shows an isosceles triangle ABC with $\overline{BA} \cong \overline{BC}$, and altitudes $\overline{CD}$ and $\overline{AE}$. By the definition of altitudes, $\overline{CD} \perp \overline{AB}$ and $\overline{AE} \perp \overline{BC}$.

We must prove that $\overline{CD} \cong \overline{AE}$. This can be done by proving △AEC ≅ △CDA and employing the corresponding parts rule of congruent triangles.

The congruent triangle postulate that can be best used in this problem is the one which states that two triangles are congruent if two angles and a side opposite one of the angles in one triangle are congruent to the corresponding parts of the other triangle. We shall refer to this rule as A.A.S. Postulate.

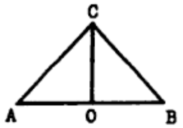
STATEMENT	REASON
1. In △ABC, $\overline{BA} \cong \overline{BC}$	1. Given.
2. ∠BAC ≅ ∠BCA	2. If two sides of a triangle are congruent, the angles opposite these sides are congruent.

3. $\overline{AE} \perp \overline{BC}$, $\overline{CD} \perp \overline{BA}$	3. Given.
4. ∡CDA and ∡AEC are right angles	4. When two perpendicular lines intersect, they form right angles.
5. ∡CDA ≅ ∡AEC	5. All right angles are congruent.
6. $\overline{AC} \cong \overline{AC}$	6. Reflexive property of congruence.
7. $\Delta ADC \cong \Delta CEA$	7. A.A.S. ≅ A.A.S.
8. $\overline{CD} \cong \overline{AE}$	8. Corresponding parts of congruent triangles are congruent.

SIDE - SIDE - SIDE POSTULATE

• PROBLEM 3-38

Prove: the median drawn to the base of an isosceles triangle bisects the vertex angle.



Solution: Draw isosceles ΔABC with median $\overline{CO}$, as in the figure shown.

We are asked to prove ∡ACO ≅ ∡BCO. By proving $\Delta ACO \cong \Delta BCO$, we can derive the desired result since the angles are corresponding angles of the two triangles.

Given: isosceles ΔABC; median $\overline{CO}$

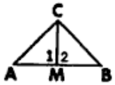
Prove: $\overline{CO}$ bisects ∡C

STATEMENTS	REASONS
1. $\overline{CO}$ is a median	1. Given.
2. $\overline{AO} \cong \overline{BO}$	2. Definition of median.
3. ΔABC is isosceles	3. Given.
4. $\overline{AC} \cong \overline{BC}$	4. Definition of an isosceles triangle.
5. $\overline{CO} \cong \overline{CO}$	5. Reflexive property.

6. $\triangle AOC \cong \triangle BOC$	6. SSS $\cong$ SSS
7. $\angle ACO \cong \angle BCO$	7. Corresponding parts of congruent triangles are congruent.
8. $\overline{CO}$ bisects $\angle C$	8. A segment that divides an angle into two angles that have equal measure bisects that angle.

• PROBLEM 3-39

Prove the median to the base of an isosceles triangle is perpendicular to the base. (In the figure, $\triangle$ ABC is isosceles, such that $\overline{CA} \cong \overline{CB}$, and $\overline{CM}$ is the median to base $\overline{AB}$. Prove $\overline{CM} \perp \overline{AB}$.)



Solution: Two methods can be used to solve this problem. The first method will utilize the theorem which states that two lines intersecting to form two congruent adjacent angles are perpendicular. The second method will employ the fact that two points each equdistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Method 1: This method involves proving $\triangle CMA \cong \triangle CMB$, and by coresponding parts, showing that $\angle 1 \cong \angle 2$. Since two adjacent angles are congruent, $\overline{CM} \perp \overline{AB}$. To prove $\triangle CMA \cong \triangle CMB$, use the S.S.S. $\cong$ S.S.S. method.

STATEMENT	REASON
1. $\overline{CA} \cong \overline{CB}$	1. Given.
2. $\overline{CM}$ is the median to base $\overline{AB}$	2. Given.
3. $\overline{AM} \cong \overline{BM}$	3. A median in a triangle divides the side to which it is drawn into two congruent parts.
4. $\overline{CM} \cong \overline{CM}$	4. Reflexive property of congruence.
5. $\triangle CMA \cong \triangle CMB$	5. S.S.S. $\cong$ S.S.S.
6. $\angle CMA \cong \angle CMB$	6. Corresponding parts of congruent triangles are congruent.

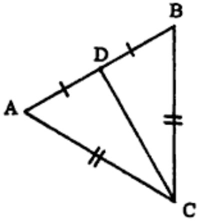
7. $\measuredangle CMA$ and $\measuredangle CMB$ are adjacent angles	7. Two angles are adjacent if they have a common vertex and a common side but do not have common interior points.
8. $\overline{CM} \perp \overline{AB}$	8. Two line segments are perpendicular if they intersect and form congruent adjacent angles.

Method 2: This method involves proving that point C and point M are both equidistant from the endpoints, A and B, of $\overline{AB}$. By the second theorem mentioned above, this will prove that $\overline{CM}$ is perpendicular to $\overline{AB}$.

STATEMENT	REASON
1. $\overline{CA} \cong \overline{CB}$ or C is equidistant from A and B	1. Given.
2. $\overline{CM}$ is a median	2. Given
3. $\overline{MA} \cong \overline{MB}$ or M is equidistant from A and B	3. A median in a triangle divides the side to which it is drawn into two congruent parts.
4. $\overline{CM} \perp \overline{AB}$	4. Two points, each equidistant from the endpoints of a line segment, determine the perpendicular bisector of the line segment.

• PROBLEM 3-40

Let ABC be an equilateral triangle. Let D be the midpoint of $\overline{AB}$. Prove that $\Delta DCB \cong \Delta DCA$. What kind of triangle is ΔDCB? What can be said about $\measuredangle ACD$ and $\measuredangle BCD$? (See figure.)



Solution: We use the S.S.S. Postulate to prove that $\Delta DCB \cong \Delta DCA$. We know that D is the midpoint of $\overline{AB}$. Hence, $\overline{DA} \cong \overline{DB}$. Also, ABC is an equilateral triangle, thus $\overline{AC} \cong \overline{BC}$.

Lastly, $\overline{DC} \cong \overline{DC}$ by the reflexive property. By the S.S.S. Postulate $\triangle DCB \cong \triangle DCA$.

Since $\triangle DCB \cong \triangle DCA$,

then
$$m\measuredangle ADC \cong m\measuredangle BDC. \tag{1}$$

But, from the figure
$$m\measuredangle ADC + m\measuredangle BDC = 180°. \tag{2}$$

Using (1) in (2) yields
$$m\measuredangle ADC = m\measuredangle BDC = 90°.$$

Hence, $\triangle DCB$ is a right triangle.

Again, because $\triangle DCB \cong \triangle DCA$,
$$\measuredangle ACD \cong \measuredangle BCD \tag{3}$$

As shown above,
$$m\measuredangle ADC = 90°. \tag{4}$$

Since $\triangle ABC$ is equilateral,
$$m\measuredangle DAC = 60°. \tag{5}$$

By the angle sum theorem,
$$m\measuredangle DAC + m\measuredangle ACD + m\measuredangle ADC = 180°. \tag{6}$$

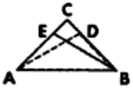
Inserting (4) and (5) in (6) gives
$$m\measuredangle ACD = 30°. \tag{7}$$

Hence, using (7) in (3)
$$m\measuredangle ACD = m\measuredangle BCD = 30°.$$

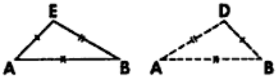
• PROBLEM 3-41

In triangle ABC, lines are drawn from points B and A to sides $\overline{AC}$ and $\overline{BC}$, respectively. As shown in the figure, two smaller triangles are formed. It is given that $\overline{CA} \cong \overline{CB}$, $\overline{CE} \cong \overline{CD}$, and $\overline{BE} \cong \overline{AD}$.

(a) Prove that $\measuredangle EAB \cong \measuredangle DBA$; (b) Find the measure of $\measuredangle EAB$, if it is represented by $5x - 8$ and the measure of $\measuredangle DBA$ equals $3x + 12$.



Separate the Triangles



Solution: (a) By proving that overlapping triangles EAB and DBA are congruent, and employing the theorem which states that corresponding parts of congruent triangles are congruent, we can show $\measuredangle EAB \cong \measuredangle DBA$. Since most of the given information concerns sides of the triangles, the logical plan of attack is to prove congruence through the S.S.S. $\cong$ S.S.S. method.

STATEMENT	REASON
1. $\overline{BE} \cong \overline{AD}$	1. Given.
2. $\overline{CA} \cong \overline{CB}$; $\overline{CE} \cong \overline{CD}$	2. Given.
3. $CA = CB$; $CE = CD$	3. Congruent segments are of equal measure.
4. $CA - CE = CB - CD$	4. Subtraction of equal quantities from equal quantities yields equals.
5. $EA = DB$	5. Substitution Postulate. Point Betweenness Postulate.
6. $\overline{EA} \cong \overline{DB}$	6. Segments of equal length are congruent.
7. $\overline{AB} \cong \overline{AB}$	7. Reflexive Property of Congruence.
8. $\Delta EAB \cong \Delta DBA$	8. S.S.S. $\cong$ S.S.S.
9. $\measuredangle EAB \cong \measuredangle DBA$	9. Corresponding parts of congruent triangles are congruent.

(b) We proved in part (a) that $\measuredangle EAB \cong \measuredangle DBA$. Thus, the measures of the two angles are equal, $m\measuredangle EAB = m\measuredangle DBA$. We have an algebraic representation for each measure and, by setting them equal to each other, we can solve for the unknown and substitute to determine the measure of $\measuredangle EAB$.

$$m\measuredangle EAB = m\measuredangle DBA$$

$$5x - 8 = 3x + 12$$

$$2x = 20$$

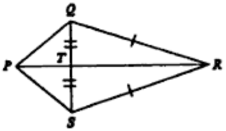
$$x = 10$$

$$m\measuredangle EAB = 5(10) - 8 = 50 - 8 = 42.$$

Therefore, the measure of $\measuredangle EAB$ is 42°.

• PROBLEM 3-42

Given: $\overline{QS}$ intersects $\overline{PR}$ at T such that RQ = RS and QT = ST.
Prove: $\overline{TP}$ bisects $\angle SPQ$.



Solution: $\overline{TP}$ bisects $\angle SPQ$ only if $\angle SPT \cong \angle QPT$. Showing $\triangle PTQ \cong \triangle PTS$ or $\triangle PQR \cong \triangle PSR$ would be sufficient to show $\angle SPT \cong \angle QPT$. Consider $\triangle PQR$ and $\triangle PSR$. $\overline{PR} \cong \overline{PR}$. $\overline{QR} \cong \overline{SR}$. To show $\triangle PQR \cong \triangle PSR$, we need one more congruence - either a pair of sides or else $\angle QRT \cong \angle SRT$.

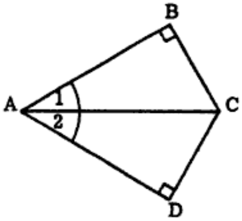
We can show $\angle QRT \cong \angle SRT$ by proving $\triangle QTR \cong \triangle STR$ by the SSS Postulate. Thus, $\triangle PQR \cong \triangle PSR$ by the SAS Postulate. $\angle QPT \cong \angle SPT$ by corresponding angles, and $\overline{TP}$ bisects $\angle QPS$.

STATEMENTS	REASONS
1. $\overline{QS}$ intersects $\overline{PR}$ at T such that RQ=RS;QT=ST	1. Given
2. $\overline{QT} \cong \overline{ST}$ $\overline{RQ} \cong \overline{RS}$	2. Two segments of equal lengths are congruent.
3. $\overline{TR} \cong \overline{TR}$	3. A segment is congruent to itself.
4. $\triangle QTR \cong \triangle STR$	4. The SSS Postulate.
5. $\angle QRT \cong \angle SRT$	5. Corresponding parts of congruent triangles are congruent.
6. $\overline{PR} \cong \overline{PR}$	6. A segment is congruent to itself.
7. $\triangle PQR \cong \triangle PSR$	7. The SAS Postulate.
8. $\angle QPR \cong \angle SPR$	8. Corresponding parts of congruent triangles are congruent.
9. $\overline{TP}$ bisects $\angle SPQ$	9. If a segment divides an angle into two congruent angles, then the segment bisects the angle.

HYPOTENUSE-LEG POSTULATE

• PROBLEM 3-43

Quadrilateral ABCD is, as shown in the figure, cut by the line $\overline{AC}$. If $\overline{CB} \perp \overline{AB}$, $\overline{CD} \perp \overline{AD}$, and $\overline{BC} \cong \overline{DC}$. Prove that $\overline{AC}$ bisects ∡BAD.



Solution: To prove $\overline{AC}$ is the bisector of ∡BAD, it is necessary to prove ∡1 ≅ ∡2. This can be done by proving ΔABC ≅ ΔADC and then using the rule which assigns congruence to corresponding parts of congruent triangles in order to prove ∡1 ≅ ∡2.

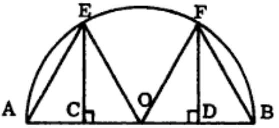
The hypotenuse leg theorem is used in this proof. It states that two right triangles are congruent if the hypotenuse and the leg of one triangle are congruent to the corresponding parts of the other triangle. Notationally, it can be written hy. leg. ≅ hy. leg.

STATEMENT	REASON
1. $\overline{CB} \perp \overline{AB}$, $\overline{CD} \perp \overline{AD}$	1. Given
2. ∡ABC and ∡ADC are right triangles	2. When perpendicular lines intersect, they form right angles.
3. Δ's ABC and ADC are right triangles	3. A triangle which contains a right angle is a right triangle.
4. $\overline{BC} \cong \overline{CD}$	4. Given.
5. $\overline{AC} \cong \overline{AC}$	5. Reflexive property of congruence.
6. ΔABC ≅ ΔADC	6. Hy. leg ≅ Hy. leg.

7. $\angle 1 \cong \angle 2$	7. Corresponding parts of congruent triangles are congruent.
8. AC bisects $\angle BAD$	8. A segment which divides an angle into two congruent angles bisects the angle.

• PROBLEM 3-44

On diameter $\overline{AB}$ of semicircle O, two points C and D are located on opposite sides of the center O so that $\overline{AC} \cong \overline{BD}$. At C and D, perpendiculars are constructed from $\overline{AB}$ and extended to meet arc $\widehat{AB}$ at points E and F, respectively. Using the fact that all radii of a semicircle are congruent, prove $\overline{CE} \cong \overline{DF}$.



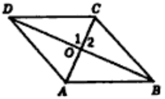
Solution: If we can prove $\Delta ECO \cong \Delta FDO$, then we will be able to conclude that $\overline{CE} \cong \overline{DF}$. Since the triangles include a right angle, they are right triangles and, as such, we can use the hypotenuse-leg theorem to prove congruence between them. Hypotenuse-leg is similar to the SSS Postulate because, when you know the hypotenuse and leg of a right triangle the third leg is uniquely determined by Pythagoras' Theorem.

STATEMENT	REASON
1. $\overline{EC} \perp \overline{AO}$ and $\overline{FD} \perp \overline{BO}$	1. Given
2. $\angle ECO$ and $\angle FDO$ are right angles	2. Perpendicular lines intersect to form right angles.
3. ΔECO and ΔFDO are right triangles	3. A triangle which contains a right angle is a right triangle.
4. $\overline{OA} \cong \overline{OB}$	4. Radii of the same circle are congruent.
5. $\overline{AC} \cong \overline{BD}$	5. Given.
6. $OA - AC = OB - BD$ or $OC = OD$ or $\overline{OC} \cong \overline{OD}$	6. Subtraction Property of Equality
7. $\overline{OE} \cong \overline{OF}$	7. Same as reason 4.

8. $\Delta ECO \cong \Delta FDO$	8. Two right triangles are congruent if the hypotenuse and leg of one triangle are congruent to the hypotenuse and leg of the second.
9. $\overline{CE} \cong \overline{DF}$	9. Corresponding sides of congruent triangles are congruent.

• PROBLEM 3-45

Given: DB and AC intersect at O. $m\angle 1 = m\angle 2 = 90°$. $\overline{CD} \cong \overline{CB}$.
Prove: $\overline{OD} \cong \overline{OB}$.



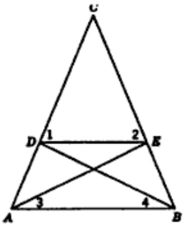
Solution: Since $\overline{OD}$ and $\overline{OB}$ are corresponding parts of right triangles COD and COB respectively, we can derive the required congruence by proving $\Delta COD \cong \Delta COB$. The hypotenuse leg theorem for congruence will be used.

STATEMENTS	REASONS
1. $m\angle 1 = m\angle 2 = 90°$	1. Given.
2. $\angle 1$ and $\angle 2$ are right angles	2. A right angle has a measure of 90°.
3. ΔCOD and ΔCOB are right triangles	3. A right triangle is a triangle with one right angle.
4. $\overline{CD} \cong \overline{CB}$	4. Given.
5. $\overline{CO} \cong \overline{CO}$	5. Reflexive property of congruence.
6. $\Delta COD \cong \Delta COB$	6. hy.leg $\cong$ hy. leg
7. $\overline{OD} \cong \overline{OB}$	7. Corresponding parts of congruent triangles are congruent.

• PROBLEM 3-46

Given: ΔABC is isosceles with base $\overline{AB}$. $\overline{BD} \perp \overline{AC}$. $\overline{AE} \perp \overline{BC}$. $\angle 1 \cong \angle 2$. Prove $\angle 3 \cong \angle 4$.

Solution: In this problem, we must first notice that $\angle 3$ and $\angle 4$ are corresponding parts of rt.Δ's AEB and BDA, respectively. Hence, by proving ΔBDA ≅ ΔAEB, we can deduce that ∡3 ≅ ∡4. The hypotenuse leg theorem for congruence will be employed.



STATEMENTS	REASONS
1. $\overline{BD} \perp \overline{AC}$; $\overline{AE} \perp \overline{BC}$	1. Given.
2. ∡BDA and ∡AEB are right angles	2. Perpendicular lines meet to form right angles.
3. ΔBDA and ΔAEB are right triangles	3. A right triangle is a triangle with one right angle.
4. ΔABC is isosceles	4. Given.
5. $\overline{AC} \cong \overline{BC}$	5. Definition of an isosceles triangle.
6. AD + DC = AC BE + EC = BC	6. The whole is congruent to the sum of its parts.
7. AD + DC = BE + EC	7. Addition Postulate.
8. ∡1 ≅ ∡2	8. Given.
9. $\overline{DC} \cong \overline{EC}$ or DC = EC	9. If a triangle has two congruent angles, then the sides opposite those angles are congruent.
10. AD = BE	10. If congruent quantities are subtracted from congruent quantities, then the differences are congruent quantities.(Steps 9 and 7).
11. AB = AB	11. Reflexive law.
12. ΔBDA ≅ ΔAEB	12. hy. leg ≅ hy. leg.
13. ∡3 = ∡4	13. Corresponding parts of congruent triangles are equal.

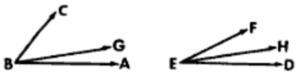
CHAPTER 4

INEQUALITIES

LINES AND ANGLES

• PROBLEM 4-1

We are given, as the figures show, $\measuredangle ABC$ and $\measuredangle DEF$. $m\measuredangle ABC > m\measuredangle DEF$. If $m\measuredangle ABG = m\measuredangle DEH$, prove $m\measuredangle GBC > m\measuredangle HEF$.



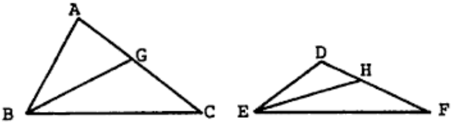
__Solution__: By realizing that when equal quantities are subtracted from unequal quantities, the differences are unequal in the same order, we can proceed directly to the proof.

Consult the diagram to clarify the steps of the proof.

__Statements__	__Reasons__
1. $m\measuredangle ABC > m\measuredangle DEF$	1. Given
2. $m\measuredangle ABG = m\measuredangle DEH$	2. Given
3. $m\measuredangle ABC - m\measuredangle ABG > m\measuredangle DEF - m\measuredangle DEH$ or $m\measuredangle GBC > m\measuredangle HEF$	3. If equal quantities are subtracted from unequal quantities, the differences are unequal in the same order.

• PROBLEM 4-2

In the accompanying figure, unequal angles ABC and DEF are bisected by $\overleftrightarrow{BG}$ and $\overleftrightarrow{EH}$, respectively. If $m\measuredangle ABC > m\measuredangle DEF$, prove $m\measuredangle ABG > m\measuredangle DEH$.



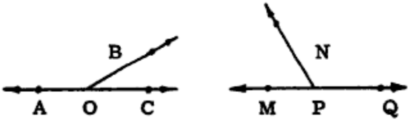
$m\angle ABC > m\angle DEF$
BG and EH bisect $\angle ABC$ and $\angle DEF$ respectively

Solution: The fact that unequal quantities remain unequal in the same order when they are divided by equal positive quantities will be employed in this proof. (Since we are dealing with an angle bisector, the specific form to be used will be that halves of unequal quantities are unequal in the same order.)

Statements	Reasons
1. $m\angle ABC > m\angle DEF$	1. Given
2. $\frac{m\angle ABC}{2} > \frac{m\angle DEF}{2}$ or $\frac{1}{2}m\angle ABC > \frac{1}{2}m\angle DEF$	2. Halves of unequal quantities are unequal in the same order.
3. $\overleftrightarrow{BG}$ bisects $\angle ABC$ $\overleftrightarrow{EH}$ bisects $\angle DEF$	3. Given
4. $m\angle ABG = \frac{1}{2}m\angle ABC$ $m\angle DEH = \frac{1}{2}m\angle DEF$	4. A bisector of an angle divides the angle into two congruent angles.
5. $m\angle ABG > m\angle DEH$	5. A quantity may be substituted for its equal in any inequality. (See steps (2) and (4).

• PROBLEM 4-3

Let $\angle AOB$ and $\angle COB$ be a linear pair and $\angle MPN$ and $\angle QPN$ be another linear pair. Prove that $m\angle AOB > m\angle MPN$ if and only if $m\angle QPN > m\angle COB$.



Solution: Two angles form a linear pair if they share a common ray, and if their second rays are directly opposite to each other. In the figure, $\angle AOB$ and $\angle COB$ are a linear pair. They share a common ray, $\overrightarrow{OB}$, and their second rays, $\overrightarrow{OA}$ and $\overrightarrow{OC}$, are directly opposite. Similarly, $\angle MPN$ and $\angle QPN$ are a linear pair.

Since the given theorem is an "if and only if" theorem, we can prove it by breaking it up into 2 parts and proving each part separately.

Part I: If $m\angle AOB > m\angle MPN$, then $m\angle QPN > m\angle COB$.

Part II: If $m\angle QPN > m\angle COB$, then $m\angle AOB > m\angle MPN$.

We proceed to prove Part I by noting that 2 angles which form a linear pair are supplementary. Since $\angle AOB$ and $\angle COB$ are a linear pair,

$$m\angle AOB + m\angle COB = 180^\circ \qquad (1)$$

Because $\angle MPN$ and $\angle QPN$ form a linear pair,

$$m\angle MPN + m\angle QPN = 180^\circ \qquad (2)$$

Equating (1) and (2) yields

$$m\angle MPN + m\angle QPN = m\angle AOB + m\angle COB$$

or

$$m\angle AOB - m\angle MPN = m\angle QPN - m\angle COB \qquad (3)$$

But, the hypothesis of Part I states that

$$m\angle AOB > m\angle MPN$$

or

$$m\angle AOB - m\angle MPN > 0 \qquad (4)$$

However, comparing (4) and (3) yields

$$m\angle QPN - m\angle COB > 0$$

or

$$m\angle QPN > m\angle COB$$

Therefore, Part I is proven. We next prove Part II.

Again, we use the fact that a linear pair of angles is supplementary. We can still use equations (1) and (2) above, because they are a statement of the fact that the angles shown in the figure are supplementary. Hence, equating (1) and (2) yields

$$m\angle MPN + m\angle QPN = m\angle AOB + m\angle COB$$

or

$$m\angle QPN - m\angle COB = m\angle AOB - m\angle MPN \qquad (5)$$

From the hypothesis of Part II,

$$m\angle QPN > m\angle COB$$

or

$$m\angle QPN - m\angle COB > 0 \qquad (6)$$

Comparing (6) and (5) yields

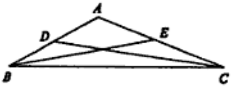
$$m\angle AOB - m\angle MPN > 0$$

or

$$m\angle AOB > m\angle MPN.$$

• **PROBLEM 4-4**

If D and E are the respective midpoints of sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$, and $AD < AE$, prove that $AB < AC$.



Solution: By the definition of midpoint, $AB = 2(AD)$ and $AC = 2(AE)$. Since we are given $AD < AE$, then, by the multiplication postulate for inequalities, $AB < AC$.

Given: $\triangle ABC$ with D and E as the respective midpoints of $\overline{AB}$ and $\overline{AC}$; $AD < AE$.

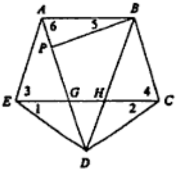
Prove: $AB < AC$

Statements	Reasons
1. $\triangle ABC$, with D and E the respective midpoints of $\overline{AB}$ and $\overline{AC}$; $AD < AE$	1. Given
2. $2(AD) < 2(AE)$	2. For all real numbers a, m, and n, if $m > n$ and $a > 0$, then $am > an$. (Multiplication Postulate for Inequality).
3. $AB = 2(AD)$; $AC = 2(AE)$	3. Definition of midpoint.
4. $AB < AC$	4. Substitution Postulate.

• PROBLEM 4-5

Given: $\measuredangle 1 \cong \measuredangle 2$; $\measuredangle 3 \cong \measuredangle 4$; $\overline{GD} \cong \overline{HD}$; $EH = CG$; $\overline{BHD}$; $\overline{EGHC}$; $\overline{APGD}$.

Prove: $m\measuredangle 5 < m\measuredangle 6$.



Solution: We can relate $\measuredangle 5$ and $\measuredangle 6$ as angles of $\triangle BAP$ or we can relate them to a third angle not in $\triangle BAP$. Too many parts of $\triangle BAP$ are unknown for the first method to be useful to us. It is difficult to see how we can show $PB > AP$ and, therefore, $m\measuredangle 6 > m\measuredangle 5$.

Using the second method, we note the symmetry of the figure: $\measuredangle 1 \cong \measuredangle 2$; $\measuredangle 3 \cong \measuredangle 4$. It would seem reasonable that $\triangle ADB$ is isosceles. If this were so, then base angles $\measuredangle 6$ and $\measuredangle ABD$ would be congruent. Since $m\measuredangle 5 < m\measuredangle ABD$, $m\measuredangle 5 < m\measuredangle 6$.

We show $\triangle ADB$ is isosceles by showing $AD = DB$. Since $AD =$

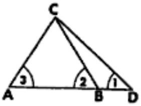
$AG + GD$, $DB = DH + HB$, and it is given that $GD = HD$, we need only show $AG = BH$. We do this by showing $\triangle AEG \cong \triangle BHC$ by the ASA Postulate.

Statements	Reasons
1. $\angle 1 \cong \angle 2$; $\angle 3 \cong 4$; $\overline{GD} \cong \overline{HD}$; $EH = CG$; $\overline{BHD}$; $\overline{EGHC}$; $\overline{APGD}$	1. Given. (We show $\triangle AEG \cong \triangle BHC$ by noting that $\angle 3 \cong \angle 4$, $\overline{EG} \cong \overline{HC}$, and $\angle AGE \cong \angle BHC$. In Step 2 to 4, we show $\angle AGE \cong \angle BHC$.)
2. $\angle HGD \cong \angle GHD$	2. The base angles of an isosceles triangle are congruent.
3. $\angle AGE \cong \angle HGD$; $\angle BHC \cong \angle GHD$	3. The opposite angles formed by intersecting lines are congruent. (Vertical Angle Theorem.)
4. $\angle AGE \cong \angle BHC$	4. Angles congruent to congruent angles are congruent. (We now show $\overline{EG} \cong \overline{HC}$ in Steps 5 to 8.)
5. $EH = EG + GH$ $CG = CH + HG$	5. Point Betweenness Postulate.
6. $EG + GH = CH + HG$	6. Substitution Postulate. (Steps 5, 1).
7. $EG = HC$	7. Subtraction Postulate.
8. $\overline{EG} \cong \overline{HC}$	8. Segments of equal length are congruent.
9. $\triangle AEG \cong \triangle BCH$	9. The ASA Postulate.
10. $\overline{AG} \cong \overline{BH}$	10. Corresponding parts of congruent triangles are congruent.
11. $AG = BH$	11. Congruent segments are of equal length.
12. $HD = GD$	12. Same reason as Step 11.
13. $AD = AG + GD$ $BD = BH + HD$	13. Definition of "betweenness."
14. $AD = BD$	14. Substitution Postulate.
15. $\angle 6 \cong \angle ABD$	15. The base angles of an isosceles triangle are congruent.
16. $m\angle 5 < m\angle ABD$	16. If point P is in the interior of $\angle ABD$, then $m\angle ABP < m\angle ABD$.
17. $m\angle 5 < m\angle 6$	17. If $a = b$ and $c < a$, then $c < b$.

EXTERIOR ANGLE THEOREM

• PROBLEM 4-6

In isosceles triangle ABC, with $\overline{AC} \cong \overline{CB}$ base $\overline{AB}$ is extended to D, and $\overline{CD}$ is drawn. Prove that $CD > CA$ (see figure).



Solution: The cornerstone of this proof is a theorem stating that the measure of an exterior angle of a triangle is greater than the measure of either remote interior angle. It can be derived from the fact that both the measure of the exterior angle, and the sum of the measures of the non-adjacent interior angles, when added to the measure of the adjacent interior angle, will sum to 180°. Since the non-adjacent angles must have positive measure, the measure of the exterior angle must be greater than either one alone.

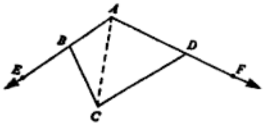
The required conclusion, $CD > CA$, will be reached by applying the theorem that if two angles of a triangle are unequal, the sides opposite these angles are unequal, and the greater side lies opposite the greater angle.

We will use the first theorem to show $m\angle 3 > m\angle 1$ and then, by the second theorem, conclude CD, opposite $\angle 3$, is greater than CA, opposite $\angle 1$.

STATEMENT	REASON
1. In $\triangle CBD$, $m\angle 2 > m\angle 1$	1. The measure of an exterior angle of a triangle is greater than the measure of either non-adjacent interior angle.
2. $\overline{AC} \cong \overline{CB}$	2. Given.
3. $\angle 3 \cong \angle 2$ or $m\angle 3 = m\angle 2$	3. Base angles of an isosceles triangle are congruent.
4. $m\angle 3 > m\angle 1$	4. A quantity may be substituted for its equal in any inequality.
5. $CD > CA$	5. If two angles of a triangle are unequal, then the sides opposite these angles are unequal, and the side of greater length lies opposite the angle of greater measure.

• PROBLEM 4-7

Given: Quadrilateral ABCD and straight rays $\overrightarrow{ADF}$ and $\overrightarrow{ABE}$.
Prove: $m\angle EBC + m\angle FDC > \frac{1}{2}(m\angle A + m\angle C)$. (Hint: Draw $\overline{AC}$)



Solution: We have made use of the hint by drawing $\overline{AC}$ in the figure shown.

Notice, that $m\angle A = m\angle BAC + m\angle CAD$ and $m\angle C = m\angle BCA + m\angle ACD$.

Since the measure of the whole is equal to the sum of the measures of its parts, we can substitute $m\angle A = m\angle BAC + m\angle CAD$ and $m\angle C = m\angle BCA + m\angle ACD$ in the equation to be proven. Hence, the equation becomes

$$m\angle EBC + m\angle FDC > \tfrac{1}{2}(m\angle BAC + m\angle CAD + m\angle BCA + m\angle ACD).$$

We will be able to derive this form of the results using the Exterior Angle Theorem. Recall that the theorem tells us that the measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

Hence, in ΔABC

$m\angle EBC > m\angle BAC$ and $m\angle EBC > m\angle BCA$.

Similarly, in ΔDAC,

$m\angle FDC > m\angle ACD$ and $m\angle FDC > m\angle CAD$.

We know that the sum of unequals are unequal in the same order. Accordingly,

$$2m\angle EBC + 2m\angle FDC > m\angle BAC + m\angle BCA + m\angle ACD + m\angle CAD.$$

Dividing by 2 and rearranging, we obtain

$$m\angle EBC + m\angle FDC > \tfrac{1}{2}(m\angle BAC + m\angle CAD + m\angle BCA + m\angle ACD).$$

These are exactly the results we set out to obtain.

TRIANGLE INEQUALITY THEOREM

• PROBLEM 4-8

If the lengths of two sides of a triangle are 10 and 14, the length of the third side may be which of the following: (a) 2 (b) 4 (c) 22 (d) 24?

Solution: By the Triangle Inequality Theorem, we know that the sum of the lengths of any two sides of a triangle must be greater than the length of the third.

Therefore, in this example, we can discover which of the lengths are possible answers by determining if the sum of the lengths of the two shortest sides is greater than the length of the longest side.

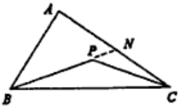
(a) 2 cannot be the third side, because $2 + 10$ is not > 14.
(b) 4 cannot be the third side, because $4 + 10$ is not > 14.
(c) 22 can be the third, because $10 + 14 > 22$.
(d) 24 cannot be the third side, because $10 + 14$ is not > 24.

Therefore, the third side may be 22.

• PROBLEM 4-9

Given: Point P is an interior point of Δ ABC.

Prove: AB + AC > BP + PC (Hint: extend $\overleftrightarrow{BP}$ so that it intersects $\overline{AC}$ at point N).



Solution: We are asked to prove an inequality involving the sides of triangles. Therefore, we apply the Triangle Inequality Theorem. This theorem states that the sum of the lengths of any two sides of a triangle is greater than the length of the third. We must now find triangles whose sides include AB, AC, BP, and PC. We use the hint and extend $\overleftrightarrow{BP}$ so that $\overleftrightarrow{BP}$ intersects $\overline{AC}$ at point N. Thus, we obtain Δ ABN and Δ PNC whose sides comprise all the lengths in question - plus an extra length PN. By the Triangle Inequality Theorem, we know in Δ ABN that AB + AN > BN or (since BN = BP + PN) that AB + AN > BP + PN. In Δ PNC, PN + NC > PC. Summing the two equations together, we obtain an inequality involving AB, AC, BP, and PC: AB + AN + NC + PN > BP + PC + PN. Combining AN + NC to obtain AC, and cancelling PN from both sides, we obtain the desired result: AB + AC > BP + PC.

Statements	Reasons
1. Point P is in the interior of Δ ABC	1. Given
2. $\overleftrightarrow{BP}$ intersects $\overline{AC}$ at point N	2. Two noncoincident, nonparallel coplanar lines intersect at a point.
3. AB + AN > BN PN + NC > PC	3. Triangle Inequality Theorem.
4. BN = PN + PB AC = AN + NC	4. Point Betweenness Postulate.
5. AB + AN + PN + NC > BN + PC	5. Addition Postulate of Inequality.
6. AB + AC + PN > PN + PB + PC	6. Substitution Postulate.
7. AB + AC > PB + PC	7. If a > b, then a - c > b - c.

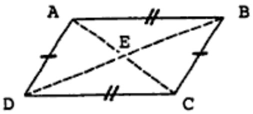
CHAPTER 5

QUADRILATERIALS

PARALLELOGRAMS

• PROBLEM 5-1

Prove that the diagonals of a parallelogram bisect each other.

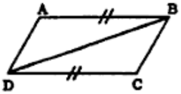


Solution: To prove that $\overline{AC}$ bisects $\overline{DB}$ (and vice versa) we must prove that $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$. We shall do this by showing that $\triangle AED \cong \triangle CEB$ (see figure).

Because ABCD is a parallelogram, opposite sides $\overline{AD} \cong \overline{CB}$. Also, $\angle DAE \cong \angle BCE$ because they are alternate interior angles for the parallel segments $\overline{AD}$ and $\overline{CB}$ cut by transversal $\overline{AC}$. Lastly, $\angle AED \cong \angle CEB$, since they are vertical angles. By the AAS Postulate, $\triangle AED \cong \triangle CEB$. Hence, by corresponding parts, $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$. From this, we conclude the diagonals of a paralellogram bisect each other.

• PROBLEM 5-2

Prove that a quadrilateral, in which one pair of opposite sides are both congruent and parallel, is a parallelogram.



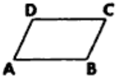
Solution: In quadrilateral ABCD (as shown in the figure) assume $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$. We shall prove that $\triangle DAB \cong \triangle BCD$ by the SAS Postulate. We will then know that $\overline{DA} \cong \overline{BC}$. Since we already know that $\overline{AB} \cong \overline{CD}$, we will have shown that both pairs of opposite sides of the quadrilateral are congruent, implying that ABCD is a parallelogram.

As given, $\overline{AB} \cong \overline{CD}$. Furthermore, since $\overline{AB} \parallel \overline{CD}$, then $\angle ABD \cong \angle CDB$. This follows because these angles are alternate interior angles of parallel segments out by a transversal. Lastly, $\overline{DB}$ is shared by both triangles, therefore $\overline{DB} \cong \overline{DB}$. By the SAS Postulate, $\triangle DAB \cong \triangle BCD$. This implies that $\overline{DA} \cong \overline{BC}$.

We have shown that both pairs of opposite sides of quadrilateral ABCD are congruent. This implies that ABCD is a parallelogram.

• PROBLEM 5-3

In parallelogram ABCD, if the measure of $\angle B$ exceeds the measure of $\angle A$ by 50°, find the measure of $\angle B$.



Solution: There is a theorem which states that the consecutive angles of a parallelogram are supplementary. This fact, more generally stated, tells us that when a transversal cuts across parallel lines the interior angles on the same side of the transversal are supplementary. In the parallelogram shown, $\overline{AD} \parallel \overline{BC}$ and $\overline{AB}$ is a transversal. Therefore, it follows that $\angle A$ and $\angle B$, being interior angles on the same side of the transversal, are supplements and their measures sum to 180°.

Hence, to find the measure of $\angle B$ (1) Let x = the measure of $\angle A$. (2) Then, $x + 50$ = the measure of $\angle B$. It follows then that

$$x + (x + 50) = 180$$

$$2x + 50 = 180$$

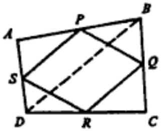
$$2x = 130$$

$$x = 65 \qquad \text{and } x + 50 = 115.$$

Therefore, the measure of $\angle B$ is 115°.

• PROBLEM 5-4

Given: P, Q, R, and S are the respective midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ of quadrilateral ABCD. Prove: Quadrilateral PQRS is a parallelogram.



Solution: There are seven different ways to show that a quadrilateral is a parallelogram. We can prove that (1) both pairs of opposite sides are parallel, (2) both pairs of opposite sides are congruent, (3) one pair of opposite sides are both parallel and congruent, (4) the opposite angles are congruent, (5) the angles of one pair of opposite angles are congruent and the sides of a pair of opposite sides are parallel,(6) the angles of one pair of opposite angles and the sides of one pair of opposite sides are congruent, or (7) the diagonals bisect each other.

Here, it is not obvious at all how to proceed. There is nothing given about the quadrilateral with which we are concerned, neither parallelism nor congruence, that can help us.

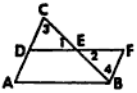
By drawing diagonal $\overline{BD}$ of ABCD, we form triangles ABD and CBD. Since sides $\overline{SP}$ and $\overline{RQ}$ of PQRS join the midpoints of two sides of ΔABD and ΔCBD respectively, they are the midlines of the triangles they lie in. According to several theorems on midlines, both $\overline{SP}$ and $\overline{RQ}$ are parallel to, and half as long as $\overline{BD}$, the side of the triangles not cross by them. By method (3),we will prove that quadrilateral PQRS is a parallelogram.

STATEMENTS	REASONS
1. P, Q, R, and S are the respective midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ of quadrilateral ABCD.	1. Given.
2. $\overline{SP}$ is a midline of ΔABD.	2. A midline of a triangle is the line segment joining the midpoints of two sides of the triangle.
3. $\overline{SP} \parallel \overline{DB}$	3. The midline of a triangle is parallel to the third side.
4. $SP = \frac{1}{2} DB$	4. The midline of a triangle is half as long as the third side of the triangle.

5. $\overline{QR}$ is a midline of ΔCDB	5. Definition of midline.
6. $\overline{QR} \parallel \overline{DB}$	6. The midline of a triangle is parallel to the third side of the triangle.
7. $QR = \frac{1}{2} DB$	7. The midline of a triangle is half as long as the third side of the triangle.
8. $\overline{SP} \parallel \overline{QR}$	8. If each of two lines is parallel to a third line, then they are parallel to each other.
9. $SP = QR$	9. Transitive property.
10. Quadrilateral PQRS is a parallelogram.	10. A quadrilateral is a parallelogram if two of its sides are both congruent and parallel.

• PROBLEM 5-5

In triangle ABC (shown in the accompanying figure), D is the midpoint of $\overline{AC}$ and E is the midpoint of $\overline{CB}$. If $\overline{DE}$ is extended to F so that $\overline{DE} \cong \overline{EF}$ and $\overline{FB}$ is drawn, prove that ABFD is a parallelogram.



Solution: To prove that ABFD is a parallelogram it is necessary to show that one pair of its opposite sides are both congruent and parallel. The fact that alternate interior angles of parallel lines cut by a transversal are congruent will be used to prove two sides are parallel. To prove congruence of the same two sides, the corresponding parts rule for congruent triangles will be employed. To derive both of the properties, congruence between Δ's BEF and CED must be established.

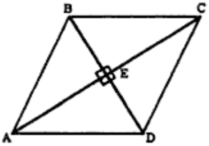
STATEMENT	REASON
1. $\overline{CE} \cong \overline{BE}$	1. Given. (E is the midpoint of $\overline{CB}$.)
2. $\overline{DE} \cong \overline{FE}$	2. Given.
3. $\measuredangle 1 = \measuredangle 2$	3. If two angles are vertical angles, then they are congruent.
4. $\Delta CED \cong \Delta BEF$	4. S.A.S. ≅ S.A.S.

5. $\measuredangle 3 \cong \measuredangle 4$	5. Corresponding parts of congruent triangles are congruent.
6. $\overline{FB} \parallel \overline{CA}$ or $\overline{FB} \parallel \overline{DA}$	6. If two lines are cut by a transversal making a pair of alternate interior angles congruent, then the lines are parallel.
7. $\overline{FB} \cong \overline{DC}$	7. Same as reason 5.
8. $\overline{DC} \cong \overline{DA}$	8. Given. (D midpoint of $\overline{CA}$.)
9. $\overline{FB} \cong \overline{DA}$	9. Transitivity of congruence.
10. ABFD is a parallelogram.	10. A quadrilateral is a parallelogram if one pair of opposite sides are both congruent and parallel.

RHOMBI

• PROBLEM 5-6

If the diagonals of a parallelogram meet at right angles, prove that the parallelogram is a rhombus.



Solution: The accompanying figure shows parallelogram ABCD, whose diagonals ($\overline{BD}$ and $\overline{AC}$)meet at right angles. By definition, a rhombus is a quadrilateral, all of whose sides are congruent. Our strategv will be to use the properties of a parallelogram to show that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$. We shall then prove that $\Delta AEB \cong \Delta AED$, which implies that $\overline{AB} \cong \overline{AD}$. Coupling these facts will show that ABCD is a quadrilateral, all of whose sides are congruent. (i.e. a rhombus).

Since ABCD is a parallelogram, it is a quadrilateral. Furthermore, by the properties of paralellograms, opposite sides are congruent. That is

$\overline{AB} \cong \overline{CD}$

and $\overline{BC} \cong \overline{DA}$. (1)

Now, focus attention on $\triangle AEB$ and $\triangle AED$. $\overline{BE} \cong \overline{DE}$, because the diagonals of a parallelogram bisect each other. $\angle BEA \cong \angle DEA$, since both are right angles. Lastly, $\overline{AE} \cong \overline{AE}$, since it is common to both triangles. By the SAS (side angle side) Postulate, $\triangle AEB \cong \triangle AED$.

Therefore, by corresponding parts

$$\overline{AB} \cong \overline{AD} \qquad (2)$$

Using (2) in (1) yields,

$$\overline{AB} \cong \overline{CD}$$

$$\overline{AB} \cong \overline{AD}$$

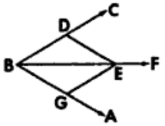
$$\overline{AD} \cong \overline{CB}$$

or $\overline{AB} \cong \overline{CD} \cong \overline{AD} \cong \overline{CB}$

which shows that ABCD is a quadrilateral, all of whose sides are congruent. Hence, ABCD is a rhombus.

• PROBLEM 5-7

In the figure shown, $\overrightarrow{BF}$ bisects angle CBA. $\overline{DE} \parallel \overrightarrow{BA}$ and $\overline{GE} \parallel \overrightarrow{BC}$. Prove GEDB is a rhombus.



Solution: We want to show that GEDB is a parallelogram which has congruent adjacent sides, since this is, by definition, a rhombus.

We will prove that GEDB is a parallelogram, and then prove $\triangle BEG \cong \triangle BED$ in order to deduce that adjacent sides $\overline{GE}$ and $\overline{DE}$ are congruent.

STATEMENT	REASON
1. $\overline{DE} \parallel \overrightarrow{BA}$, $\overline{GE} \parallel \overrightarrow{BC}$	1. Given.
2. GEDB is a parallelogram	2. A quadrilateral in which all opposite sides are parallel is a parallelogram.
3. $\overrightarrow{BF}$ is the angle bisector of $\angle CBA$	3. Given.

4. $\angle DBE \cong \angle GBE$	4. An angle bisector divides the angle into two congruent angles.
5. $\overline{BG} \cong \overline{DE}$	5. Opposite sides of a parallelogram are congruent.
6. $\overline{BE} \cong \overline{BE}$	6. Reflexive Property of Congruence.
7. $\Delta BEG \cong \Delta BED$	7. S.A.S. $\cong$ S.A.S.
8. $\overline{GE} \cong \overline{DE}$	8. Corresponding sides of congruent triangles are congruent.
9. GEDB is a rhombus	9. A rhombus is a parallelogram with adjacent sides congruent.

• PROBLEM 5-8

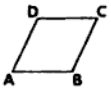
As drawn in the figure, ABCD is a parallelogram. The lengths of each side can be represented in the following algebraic manner:

$$AB = 2x + 1$$
$$DC = 3x - 11$$
$$AD = x + 13$$

Show that ABCD is a rhombus.

Solution: A rhombus is a quadrilateral in which all sides are congruent. Since ABCD is given to be a parallelogram we know it is a quadrilateral whose opposite sides are congruent. Therefore, to prove ABCD is a rhombus, it is sufficient to show that any two consecutive sides are congruent. (Any congruent sides are of equal length.)

In effect, we are given $DC \cong AB$ and want to show $AD \cong AB$.



Set the length of $\overline{AB}$ equal to the length of $\overline{DC}$ and solve for x. Then, substitute back into the expressions for the length of $\overline{AB}$ and $\overline{AD}$ to determine whether they are equal.

$$DC = AB$$

$$3x - 11 = 2x + 1$$

$$3x - 2x = 1 + 11$$

$$x = 12.$$

Then, $AB = 2x + 1 = 2(12) + 1 = 25$

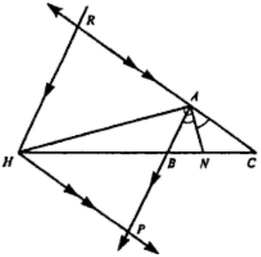
$AD = x + 13 = 12 + 13 = 25.$

Therefore, the length of $\overline{AB}$ = length of $\overline{AD}$ and $\overline{AB} \cong \overline{AD}$.

Therefore, ABCD is a rhombus because it is a quadrilateral (a parallelogram) in which two consecutive sides are congruent.

• PROBLEM 5-9

Given: $\overline{AN}$ is an angle bisector of $\triangle ABC$; $\overleftrightarrow{CNBH}$; $\overline{HA} \perp \overline{AN}$; $\overrightarrow{CAR}$; $\overrightarrow{HR} \parallel \overline{AB}$; $\overrightarrow{ABP}$; $\overrightarrow{HP} \parallel \overline{CA}$. Prove: quadrilateral APHR is a rhombus.



Solution: Aside from the various collinearities, we are given: (1) $\overrightarrow{HP} \parallel \overline{CA}$; $\overrightarrow{HR} \parallel \overline{AB}$; (2) $\overline{AN}$ is an angle bisector of $\triangle ABC$; and (3) $\overline{HA} \perp \overline{AN}$. From (1), we can show that quadrilateral APHR has both pairs of opposite sides parallel, and therefore APHR is a parallelogram.

Given that a figure is a parallelogram, we can show that it is a rhombus if we show that a diagonal of the parallelogram bisects an angle of the parallelogram. Most of the information we are given involves vertex A. Therefore, we prove the result by showing diagonal $\overline{AH}$ bisects $\measuredangle RAP$ or $m\measuredangle RAH = m\measuredangle HAP$; we set out finding expressions for $m\measuredangle RAH$ and $m\measuredangle HAP$; and try to show them equal.

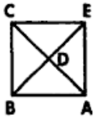
STATEMENTS	REASONS
1. $\overline{AN}$ is an angle bisector of $\triangle ABC$; $\overleftrightarrow{CNBH}$; $\overline{HA} \perp \overline{AN}$; $\overrightarrow{CAR}$; $\overrightarrow{HR} \parallel \overline{AB}$; $\overrightarrow{ABP}$; $\overrightarrow{HP} \parallel \overline{CA}$	1. Given.

2. $\overline{RA} \parallel \overline{HP}$; $\overline{HR} \parallel \overline{AP}$	2. Two segments each on separate parallel lines are parallel.
3. ▱APHR is a paralleogram	3. If both pairs of opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.
4. ∢HAN is a right angle or m∢HAN = 90°	4. Perpendicular lines intersect to form right angles.
5. ∢HAB and ∢BAN are complementary	5. Two angles that form a right angle are complementary.
6. m∢RAC = m∢RAH + m∢HAN + m∢NAC	6. Angle Sum Postulate.
7. m∢RAC = 180°	7. The measure of a straight angle is 180°.
8. 180° = m∢RAH + 90° + m∢NAC	8. Substitution Postulate.
9. 90° = m∢RAH + m∢NAC	9. Subtraction Postulate.
10. ∢RAH and ∢NAC are complementary	10. Two angles whose measures sum to 90° are complementary angles.
11. ∢NAC ≅ ∢BAN	11. Definition of angle bisector.
12. ∢RAH ≅ ∢HAB	12. Angles complementary to congruent angles are congruent.
13. $\overline{HA}$ bisects ∢RAP	13. Definition of angle bisector.
14. Quadrilateral APHR is a rhombus	14. If the diagonal of a parallelogram bisects an angle of the parallelogram, then the parallelogram is a rhombus.

SQUARES

• PROBLEM 5-10

In the accompanying figure, △ABC is given to be an isosceles right triangle with ∢ABC a right angle and AB ≅ BC. Line segment $\overline{BD}$, which bisects $\overline{CA}$, is extended to E, so that $\overline{BD}$ ≅ $\overline{DE}$. Prove BAEC is a square.

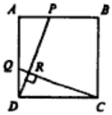


Solution: A square is a rectangle in which two consecutive sides are congruent. This definition will provide the framework for the proof in this problem. We will prove that BAEC is a parallelogram that is specifically a rectangle with consecutive sides congruent, namely a square.

STATEMENT	REASON
1. $\overline{BD} \cong \overline{DE}$ and $\overline{AD} \cong \overline{DC}$	1. Given ($\overline{BD}$ bisect $\overline{CA}$)
2. BAEC is a parallelogram	2. If diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
3. $\angle ABC$ is a right angle	3. Given.
4. BAEC is a rectangle	4. A parallelogram, one of whose angles is a right angle, is a rectangle.
5. $\overline{AB} \cong \overline{BC}$	5. Given.
6. BAEC is a square	6. If a rectangle has two congruent consecutive sides, then the rectangle is a square.

• PROBLEM 5-11

Given: Square ABCD; P is any point of $\overline{AB}$, Q is any point on $\overline{AD}$, $\overline{CQ} \perp \overline{PD}$ at R. Prove $\overline{PD} \cong \overline{QC}$.



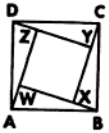
Solution: $\overline{PD}$ and $\overline{QC}$ are corresponding parts of ΔAPD and ΔDQC. It is then sufficient to show ΔAPD and ΔDQC are congruent. We do this using the AAS Postulate.

STATEMENTS	REASONS
1. Square ABCD; P is any point of $\overline{AB}$ so that $\overline{APB}$; $\overline{CQ} \perp \overline{PD}$ at R; $\overline{AQD}$	1. Given.

2. ∢DAP and ∢ADC are right angles.	2. All angles of a square are right angles.
3. ∢DAP ≅ ∢ADC	3. All right angles are congruent.
4. $\overline{AD} \cong \overline{DC}$	4. All sides of a square are congruent.
5. ∢APD and ∢ADP are complementary	5. The acute angles of a right triangle (ΔDAP) are complementary
6. ∢DQR and ∢ADP are complementary	6. The acute angles of a right triangle (ΔQDR) are complementary
7. ∢APD ≅ ∢DQR	7. Angles complementary to congruent angles are congruent.
8. ΔAPD ≅ ΔDQC	8. The AAS Postulate.
9. $\overline{PD} \cong \overline{QC}$	9. Corresponding parts of congruent triangles are congruent.

• PROBLEM 5-12

In the accompanying figure, WXYZ is a square. It has been placed in such a way inside quadrilateral ABCD that when its sides are extended to the vertices of ABCD, $\overline{AW} \cong \overline{BX} \cong \overline{CY} \cong \overline{DZ}$. All segments shown are straight line segments. Show that ABCD is a square.



Solution: In this problem we will set out to show Δ's AWB, BXC, CYD, and DZA are congruent. Then, by several applications of the fact that corresponding parts of congruent triangles are congruent, we will show that ABCD is a quadrilateral all of whose sides are congruent, and which contains a right angle (i.e., ABCD is a square).

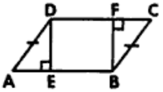
STATEMENT	REASON
1. WXYZ is a square	1. Given.
2. $\overline{AW} \cong \overline{BX} \cong \overline{CY} \cong \overline{DZ}$	2. Given.

3. $\overline{WZ} \cong \overline{XW} \cong \overline{YX} \cong \overline{ZY}$	3. All four sides of a square are congruent.
4. $\overline{AW} + \overline{WZ} \cong \overline{BX} + \overline{XW}$ $\overline{CY} + \overline{YX} \cong \overline{DZ} + \overline{ZY}$ or $\overline{AZ} \cong \overline{BW} \cong \overline{CX} \cong \overline{DY}$	4. Congruent quantities added to congruent quantities result in congruent quantities.
5. $\overline{DZ} \cong \overline{AW} \cong \overline{BX} \cong \overline{CY}$	5. Given.
6. $\overleftrightarrow{BW}$, $\overleftrightarrow{CX}$, $\overleftrightarrow{DY}$ and $\overleftrightarrow{AZ}$ are straight lines	6. Given.
7. $\overline{BW} \perp \overline{AZ}$, $\overline{AZ} \perp \overline{DY}$, $\overline{DY} \perp \overline{CX}$ and $\overline{CX} \perp \overline{BW}$	7. The sides of a square meet to form right angles and are, therefore, perpendicular.
8. $\measuredangle AZD$, $\measuredangle AWB$, $\measuredangle CXB$ and $\measuredangle CYD$ are right angles	8. Perpendicular lines intersect to form right angles.
9. $\measuredangle AZD \cong \measuredangle AWB \cong \measuredangle CXB \cong \measuredangle CYD$	9. All right angles are congruent.
10. $\Delta AZD \cong \Delta BWA \cong \Delta CXB \cong \Delta DYC$	10. SAS $\cong$ SAS.
11. $\overline{AD} \cong \overline{DC} \cong \overline{CB} \cong \overline{BA}$	11. Corresponding sides of congruent triangles are congruent.
12. $\measuredangle WAB$ is complementary to $\measuredangle WBA$	12. The acute angles of right triangles are complementary.
13. $\measuredangle ZAD \cong \measuredangle WBA$	13. Corresponding parts of congruent triangles are congruent.
14. $\measuredangle WAB$ is complementary to $\measuredangle ZAD$	14. An angle congruent to a given angle's complement is complementary to that given angle.
15. $m\measuredangle WAB + m\measuredangle ZAD = 90°$	15. The sum of the measures of two complements is 90°.
16. $m\measuredangle DAB = m\measuredangle WAB + m\measuredangle ZAD$	16. The measure of the whole is equal to the sum of the measures of its parts.
17. $m\measuredangle DAB = 90°$	17. Substitution Postulate.
18. $\measuredangle DAB$ is a right angle	18. An angle that measures 90° is a right angle.
19. ABCD is a square	19. A quadrilateral that contains a right angle and all of whose sides are congruent is a square.

RECTANGLES

• PROBLEM 5-13

In parallelogram ABCD, as shown in the figure, $\overline{DE} \perp \overline{AB}$ and $\overline{BF} \perp \overline{DC}$. With this in mind, prove altitudes $\overline{DE} \cong \overline{BF}$.



Solution: To prove that $\overline{DE} \cong \overline{BF}$, the best approach would be to prove that the triangles, which have $\overline{DE}$ and $\overline{BF}$ as corresponding parts, are congruent. Therefore, we want to prove that $\triangle AED \cong \triangle CFB$. The method of proving congruence that is best for this problem is the one in which two triangles are shown to be congruent because two angles and the side opposite one angle in one triangle are congruent to the corresponding parts of another triangle. (A.A.S. $\cong$ A.A.S.)

Additionally, to prove the parts congruent it is essential to remember that opposite sides and angles in a parallelogram are congruent.

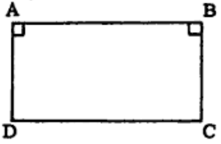
STATEMENT	REASON
1. ABCD is a parallelogram.	1. Given.
2. $\overline{AD} \cong \overline{CB}$	2. Opposite sides of a parallelogram are congruent.
3. $\angle A \cong \angle C$	3. Opposite angles of a parallelogram are congruent.
4. $\overline{DE} \perp \overline{AB}$ and $\overline{BF} \perp \overline{DC}$	4. Given.
5. $\angle E$ and $\angle F$ are right angles.	5. When perpendicular lines intersect they form right angles.
6. $\angle E \cong \angle F$	6. All right angles are congruent.
7. $\triangle AED \cong \triangle CFB$	7. a.a.s. $\cong$ a.a.s.
8. $\overline{DE} \cong \overline{BF}$	8. Corresponding parts of congruent triangles are congruent.

• PROBLEM 5-14

Prove that a rectangle is a parallelogram.

Solution: A rectangle is a quadrilateral, all of whose angles measure 90° (see figure). We will show that ABCD is a parallelogram by analyzing the pairs of angles ABC and BCD, and DAB and ABC.

Since $\angle ABC$, $\angle BCD$, and $\angle DAB$ are all right angles,



$$m\angle ABC = 90° \qquad (1)$$

$$m\angle BCD = 90° \qquad (2)$$

$$m\angle DAB = 90° \qquad (3)$$

Adding (1) and (2) shows that $\angle ABC$ and $\angle BCD$ are supplementary:

$$m\angle ABC + m\angle BCD = 180°. \qquad (4)$$

Adding (1) and (3) shows that $\angle ABC$ and $\angle DAB$ are supplementary:

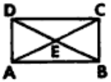
$$m\angle ABC + m\angle DAB = 180°. \qquad (5)$$

Now, if the consecutive interior angles of 2 lines crossed by a transversal sum to 180°, then the 2 lines are parallel.

$\angle ABC$ and $\angle BCD$ are consecutive interior angles of line segments $\overline{AB}$ and $\overline{DC}$. Also, $\angle ABC$ and $\angle DAB$ are consecutive interior angles of line segments $\overline{AD}$ and $\overline{BC}$. Using these facts, plus (4) and (5), we conclude that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$. By definition this means that ABCD, a rectangle, is a parallelogram.

• PROBLEM 5-15

In the figure shown, we are given rectangle ABCD. (a) Prove $\overline{AC} \cong \overline{BD}$. (b) Prove $\triangle AEB$ is isosceles.



Solution: (a) In this part, we are asked to prove that the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent. Since the only given fact is that ABCD is a rectangle, we are proving that, in general, the diagonals of a rectangle are congruent.

We will do this by proving $\triangle ACB \cong \triangle BDA$, using the SAS Postulate.

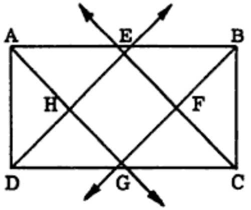
STATEMENT	REASON
1. ABCD is a rectangle	1. Given.
2. $\overline{AD} \cong \overline{BC}$	2. Opposite sides of a rectangle are congruent.
3. $\overline{AB} \cong \overline{BA}$	3. Reflexive Property of Congruence.
4. $\angle DAB$ and $\angle CBA$ are right angles	4. All angles of a rectangle are right angles.
5. $\angle DAB \cong \angle CBA$	5. All right angles are congruent.
6. $\triangle ACB \cong \triangle BDA$	6. SAS $\cong$ SAS
7. $\overline{AC} \cong \overline{BD}$	7. Corresponding sides of congruent triangles are congruent.

(b) To prove $\triangle AEB$ isosceles, we must prove that $\overline{AE} \cong \overline{EB}$. We can do this by showing that the angles opposite these sides are congruent.

STATEMENT	REASON
1. $\triangle ACB \cong \triangle BDA$	1. Results from part (a).
2. $\angle CAB \cong \angle DBA$	2. Corresponding angles of congruent triangles are congruent.
3. In $\triangle AEB$, $\overline{AE} \cong \overline{EB}$	3. In a triangle, if two angles are congruent, then the sides opposite those angles are congruent.
4. $\triangle AEB$ is isosceles	4. By definition, if a triangle has a pair of congruent sides, then it is isosceles.

• PROBLEM 5-16

Prove that the bisectors of the angles of a rectangle enclose a square.



Solution: In the diagram, the angle bisectors of rectangle ABCD intersect at points E, F, G, and H. We must show that quadrilateral EFGH is a square. To do this we will show that (1) EFGH is a rectangle and (2) two adjacent sides of EFGH are congruent.

To show that EFGH is a rectangle, we show that it is a quadrilateral with four right angles. We will do this with several applications of the theorem stating that the angle sum of a triangle is 180°.

$\overline{AG}$ is an angle bisector of $\angle DAB$. Since quadrilateral ABCD is a rectangle, then $\angle DAB$ is a right angle. Thus, $\angle DAH = \frac{1}{2}(90^\circ) = 45^\circ$. Similarly, $\overline{DE}$ is an angle bisector of $\angle ADC$, and thus $\angle HDA = \frac{1}{2}(90^\circ) = 45^\circ$. In $\triangle HDA$, $m\angle AHD + m\angle HDA + m\angle DAH = 180^\circ$, or $m\angle AHD = 180^\circ - m\angle DAH - m\angle HDA = 180^\circ - 45^\circ - 45^\circ = 90^\circ$. Because $\angle EHG$ and $\angle AHD$ are opposite angles, $m\angle EHG = m\angle AHD = 90^\circ$. By similar reasoning, we can show $m\angle EFG = 90^\circ$.

To show $m\angle DEC$ is 90°, note that $m\angle HDG = \frac{1}{2}m\angle ADG = \frac{1}{2}(90^\circ) = 45^\circ$. Also, $m\angle ECD = \frac{1}{2}\, m\angle BCD = \frac{1}{2}(90^\circ) = 45^\circ$. In $\triangle DEC$, $m\angle HDG + m\angle ECD + m\angle DEC = 180^\circ$. $m\angle DEC = 180^\circ - m\angle EDC - m\angle ECD = 180^\circ - 45^\circ - 45^\circ = 90^\circ$. By analogous reasoning we can show $m\angle AGB = 90^\circ$.

Since all four vertex angles are right angles, quadrilateral EFGH is a rectangle. The proof of the second part - that two adjacent sides of EFGH are congruent - follows:

(1) DE = EC:	Note $m\angle EDC = m\angle ECD = 45^\circ$. Thus, $\triangle DEC$ is isosceles and DE = EC.
(2) $\triangle AHD \cong \triangle BFC$:	$m\angle DAH = m\angle CBF = 45^\circ$. $m\angle ADH = m\angle BCF = 45^\circ$. Opposite sides of rectangle ABCD are congruent. Thus, $\overline{AD} \cong \overline{BC}$. By the ASA Postulate, $\triangle AHD \cong \triangle BFC$.
(3) HD = FC:	Corresponding parts of congruent triangles are congruent and equal in length.

(4) DE - HD =
EC - FC : By the Subtraction Property of Equality.

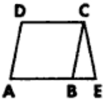
(5) HE = EF: Substitution Postulate.

Thus, in rectangle EFGH, two adjacent sides HE and EF are congruent. Therefore, EFGH is a square.

TRAPEZOIDS

• PROBLEM 5-17

We are given parallelogram ABCD and straight line segment $\overline{AE}$. (See the figure.) $\measuredangle CBE \cong \measuredangle CEB$. Present a formal proof to show AECD is an isosceles trapezoid.



Solution: A trapezoid is a quadrilateral that has two and only two sides parallel. It is isosceles if the two non-parallel sides are congruent.

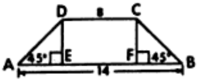
In this problem, we will first prove AECD is a trapezoid. Then, $\overline{DA}$ will be shown to be congruent to $\overline{CE}$ by first proving that both of the former are congruent to the same segment and then applying the transitive property of congruence.

STATEMENT	REASON
1. ABCD is a parallelogram	1. Given.
2. $\overline{DC} \parallel \overline{AB}$ (or $\overline{DC} \parallel \overline{AE}$)	2. Opposite sides of a parallelogram are parallel.
3. $\overline{DA} \parallel \overline{CB}$	3. Same as reason 2.
4. $\overline{DA} \nparallel \overline{CE}$	4. Through a point not on a given line only one line can be drawn parallel to that given line.
5. AECD is a trapezpoid	5. A trapezoid is a quadrilateral that has two and only two sides parallel.
6. $\overline{DA} \cong \overline{CB}$	6. A pair of opposite sides of a parallelogram are congruent, and statement 1.

7. $\angle CBE \cong \angle CEB$	7. Given.
8. $\overline{CB} \cong \overline{CE}$	8. If two angles of a triangle are congruent, then the sides opposite those angles are congruent.
9. $\overline{DA} \cong \overline{CE}$	9. Transitive property of congruence.
10. AECD is an isosceles trapezoid	10. An isosceles trapezoid is a trapezoid whose non-parallel sides are congruent.

• PROBLEM 5-18

The lengths of the bases of an isosceles trapezoid are 8 and 14, and each of the base angles measures 45°. Find the length of the altitude of the trapezoid.



Solution: As can be seen in the figure, it is helpful to draw both altitudes $\overline{DE}$ and $\overline{CF}$. DCFE is a rectangle because $\overline{DC} \parallel \overline{AB}$ (given by the definition of a trapezoid), $\overline{DE} \parallel \overline{CF}$ (they are both $\perp$ to the same line) and $\angle E$ is a right angle ($\overline{DE} \perp \overline{AB}$). Opposite sides of a rectangle are congruent. Therefore, $DC = EF = 8$.

$\Delta AED \cong \Delta BFC$ because $\angle DEA \cong \angle CFB$ (both are right angles).

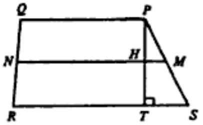
$\angle DAE \cong \angle CBF$, and $\overline{DA} \cong \overline{BC}$. (The last two facts come from the definition of an isosceles trapezoid). Therefore, by corresponding parts, $\overline{AE} \cong \overline{BF}$. Then, $AE = \frac{1}{2}(AB - EF)$ and, substituting $AE = \frac{1}{2}(14 - 8) = \frac{1}{2}(6) = 3$.

In ΔAED, $m\angle E = 90$ and $m\angle A = 45$. By the angle sum postulate for triangles, $m\angle D = 45$. Since ΔAED has two angles of equal measure, it must be an isosceles triangle. Therefore, $AE = DE = 3$.

Therefore, the length of the altitude of trapezoid ABCD is 3.

• PROBLEM 5-19

Prove that the line containing the median of a trapezoid bisects any altitude of the trapezoid.



Solution: The median of a trapezoid is the line segment joining the midpoints of the nonparallel sides. Since the median is everywhere half way between the bases, it is everywhere equidistant from the two bases and, as such, is parallel to them. The nonparallel sides and any altitude of a trapezoid are all transversals of the parallel bases and median.

There is a theorem that states if three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal. Since parallel lines $\overleftrightarrow{QP}$, $\overleftrightarrow{NM}$, and $\overleftrightarrow{RS}$ cut equal segments on $\overline{QR}$ and $\overline{PS}$, they cut $\overline{PT}$ equally.

Given: Trapezoid PQRS with altitude $\overline{PT}$ intersecting median $\overline{MN}$ at H.

Prove: $\overline{MN}$ bisects $\overline{PT}$.

STATEMENTS	REASONS
1. Trapezoid PQRS with altitude $\overline{PT}$ meeting median $\overline{MN}$ at H.	1. Given.
2. $\overline{MN} \parallel \overline{SR}$	2. The median of a trapezoid is parallel to the bases.
3. M is the midpoint of $\overline{PS}$	3. Definition of the median of a triangle.
4. $\overline{PM} \cong \overline{MS}$	4. Definition of the midpoint of a line segment.
5. $\overline{PH} \cong \overline{HT}$	5. If three or more parallel lines intercept congruent segments on a transversal, then they intercept congruent segments on any other transversal.
6. $\overline{MN}$ bisects $\overline{PT}$	6. Definition of bisection.

CHAPTER 6

GEOMETRIC PROPORTIONS AND SIMILARITIES

RATIO AND PROPORTIONS

• PROBLEM 6-1

Solve for the unknown, c, in the proportion 18 : 6 = c : 9.

Solution: A theorem tells us that, in a proportion, the product of the means is equal to the product of the extremes.

In this problem, 6 and c are the means and 18 and 9 are the extremes. Therefore,

$$6c = 18 \times 9$$

$$6c = 162$$

$$c = \frac{162}{6} = 27.$$

The answer c = 27 can be checked by substituting back into the original proportion.

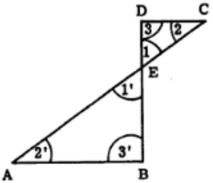
$$18 : 6 = 27 : 9$$

$$3 : 1 = 3 : 1$$

Since the two ratios are equal, c = 27.

• PROBLEM 6-2

Line segments $\overline{AC}$ and $\overline{BD}$ intersect at E, as shown in the accompanying diagram. $\overline{AB}$ is parallel to $\overline{CD}$. (a) Prove that $\Delta ABE \sim \Delta CDE$ and (b) if DE = 10, BE = 15 and CE = 20, find AE.



Solution: (a) For this proof, we will employ the theorem which tells us that two triangles are similar if two angles of one triangle are congruent to two corresponding angles of the other.

STATEMENT	REASON
1. $\overline{AC}$ and $\overline{BD}$ are straight line segments	1. Given.
2. $\angle 1$ and $\angle 1'$ are vertical angles	2. Definition of vertical angles
3. $\angle 1 \cong \angle 1'$	3. If two angles are vertical angles, then they are congruent.
4. $\overline{AB} \parallel \overline{CD}$	4. Given.
5. $\angle 2 \cong \angle 2'$	5. If parallel lines are cut by a transversal, then the alternate interior angles are congruent.
6. $\Delta ABE \sim \Delta CDE$	6. A.A. Similarity Theorem.

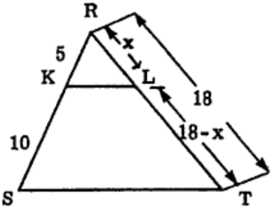
(b) Corresponding sides of similar triangles must be proportional. Therefore, since DE corresponds to BE and CE corresponds to AE, the proportion DE : BE = CE : AE must hold. The lengths of all sides other than $\overline{AE}$ are known and can be substituted into the proportion, enabling us to calculate AE.

$$10 : 15 = 20 : AE$$

$$10\,(AE) = 15 \cdot 20$$

$$AE = 30.$$

In the accompanying figure, the line segment, $\overline{KL}$, is drawn parallel to $\overline{ST}$, intersecting $\overline{RS}$ at K and $\overline{RT}$ at L in ΔRST. If RK = 5, KS = 10, and RT = 18, then find RL.



Solution: If a line is parallel to one side of a triangle, then it divides the other two sides proportionally. Since $\overline{KL} \parallel \overline{ST}$, $\overline{RT}$ and $\overline{RS}$ are divided proportionally.

Let x = RL. Then 18 - x = LT. Set up the proportion

$\frac{RK}{KS} = \frac{RL}{LT}$ and substitute to obtain

$$\frac{5}{10} = \frac{x}{18 - x}$$

or $10x = 90 - 5x$

$$15x = 90$$

$$x = 6$$

Therefore, x = RL = 6.

Alternatively, instead of forming a ratio of upper to lower segment, we can form a ratio of the upper segment to the whole side.

We are given RT = 18, RS = 15, and RK = 5. If we let RL = x, the proportion becomes

$$\frac{RK}{RS} = \frac{RL}{RT}$$

$$\frac{5}{15} = \frac{x}{18}$$

$$15x = 90$$

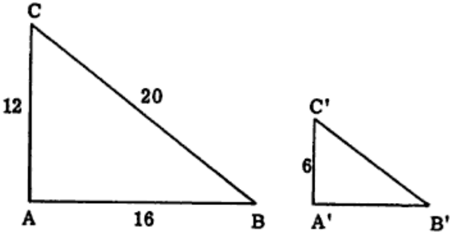
$$x = 6.$$

It is seen, then, that the same solution can be arrived at in several ways.

PROVING TRIANGLES SIMILAR

• PROBLEM 6-4

In the accompanying figure, triangle ABC is similar to triangle A'B'C', and AC corresponds to A'C'. (a) Find the ratio of similitude. (b) Find A'B' and B'C'.



Solution: (a) The ratio of similitude of two similar polygons is defined as the ratio of the measures of any two corresponding sides. We are told AC corresponds to A'C'. Therefore,

$$\text{Ratio of Similitude} = \frac{AC}{A'C'} = \frac{12}{6} = \frac{2}{1} = 2 : 1.$$

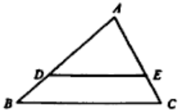
(b) The ratio of similitude of ΔABC to $\Delta A'B'C'$ being 2 : 1 tells us that each side of ΔABC is twice as long as its corresponding part in $\Delta A'B'C'$. Conversely, each side of $\Delta A'B'C'$ is ½ as long as its corresponding side in ΔABC.

(i) $A'B' = \frac{1}{2}(AB) = \frac{1}{2}(16) = 8.$

(ii) $B'C' = \frac{1}{2}(BC) = \frac{1}{2}(20) = 10.$

• PROBLEM 6-5

Let ABC be a triangle where D is a point on $\overline{AB}$, and E is a point on $\overline{AC}$. Prove that if $\overline{DE} \parallel \overline{BC}$, then AB/AD = BC/DE. (See figure.)



Solution: We will use the fact that $\overline{DE} \parallel \overline{BC}$ to prove that $\Delta ADE \sim \Delta ABC$. We can then set up a proportion and show that AB/AD = BC/DE.

Since the corresponding angles of 2 parallel lines cut by a transversal are congruent, $\sphericalangle ADE \cong \sphericalangle ABC$ and $\sphericalangle AED \cong \sphericalangle ACB$. Furthermore, $\sphericalangle DAE \cong \sphericalangle BAC$. Therefore, by the A.A.A. (angle angle angle) Similarity Theorem, $\Delta ADE \sim \Delta ABC$. Hence, corresponding sides of the 2 triangles must be proportional, and we may write

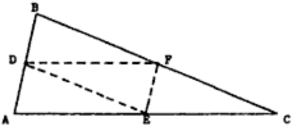
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} \qquad \text{or} \qquad \frac{AD}{AB} = \frac{DE}{BC} .$$

Taking reciprocals of both sides:

$$\frac{AB}{AD} = \frac{BC}{DE} .$$

• PROBLEM 6-6

(a) In ΔABC, if D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$, and F is the midpoint of $\overline{BC}$, then prove that $\overline{DE} \parallel \overline{BC}$, $\overline{EF} \parallel \overline{AB}$, and $\overline{DF} \parallel \overline{AC}$. (b) Prove that $\Delta DEF \sim \Delta CBA$.



Solution: (a) The method is to show that $\Delta ADE \sim \Delta ABC$. We will then have $\sphericalangle ADE \cong \sphericalangle ABC$ and be able to conclude that $\overline{DE} \parallel \overline{BC}$. The same procedure is used for the pairs of triangles CFE and CBA, and BDF and BAC.

Focus attention on triangles ADE and ABC.

First, D and E are the midpoints of $\overline{BA}$ and $\overline{CA}$, respectively. Hence,

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$$

Also, $\sphericalangle DAE \cong \sphericalangle BAC$. Therefore, by the Side Angle Side (S.A.S.) Similarity Theorem, $\Delta ADE \sim \Delta ABC$. By definition then, $\sphericalangle ADE \cong \sphericalangle ABC$. Because these 2 angles are congruent corresponding angles of the segments $\overline{DE}$ and $\overline{BC}$, $\overline{DE} \parallel \overline{BC}$.

In an exactly analogous manner, we may prove that $\Delta CFE \sim \Delta CBA$, and $\Delta BDF \sim \Delta BAC$. Hence, $\sphericalangle CFE \cong \sphericalangle CBA$ and

$\angle BDF \cong \angle BAC$. Therefore, $\overline{FE} \parallel \overline{BA}$ and $\overline{DF} \parallel \overline{AC}$.

(b) From part (a), we know that

$$\begin{aligned} \Delta ADE &\sim \Delta ABC \\ \Delta CFE &\sim \Delta CBA \qquad (1) \\ \Delta BDF &\sim \Delta BAC. \end{aligned}$$

Hence, from equations (1),

$$\begin{aligned} \frac{DE}{BC} &= \frac{AD}{AB} = \frac{1}{2} \\ \frac{FE}{BA} &= \frac{CF}{CB} = \frac{1}{2} \qquad (2) \\ \frac{DF}{AC} &= \frac{BD}{BA} = \frac{1}{2} \end{aligned}$$

Here, we have used the fact that D, E, and F are midpoints of $\overline{BA}$, $\overline{AC}$, and $\overline{CB}$, respectively. From (2), then,

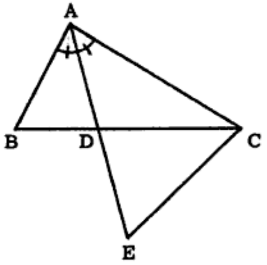
$$\frac{DE}{BC} = \frac{FE}{BA} = \frac{DF}{AC} = \frac{1}{2}$$

By the S.S.S. (Side Side Side) Similarity Theorem,

$$\Delta DEF \sim \Delta CBA.$$

• PROBLEM 6-7

Given: $\overline{AD}$ is an angle bisector of ΔABC; point E is on $\overleftrightarrow{AD}$ such that $AB \cdot AC = AD \cdot AE$. Prove: $\angle B \cong \angle AEC$.

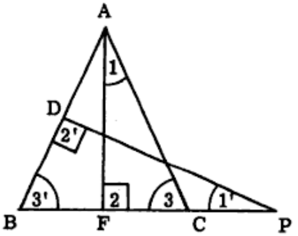


Solution: Whenever there are proportions, the best procedure is to find similar triangles. There are five possible triangles in the figure but only two, ΔABD and ΔAEC, allow us to use the given fact that $\measuredangle BAD \cong \measuredangle EAC$. By using this and the information that $AB \cdot AC = AD \cdot AE$ (which show that two pairs of sides are proportional) we can prove $\triangle ABD \sim \triangle AEC$ by the A.S.A. Similarity Theorem. The congruence of $\measuredangle B$ and $\measuredangle AEC$ follows immediately.

STATEMENT	REASONS
1. $\overline{AD}$ bisects $\measuredangle BAC$	1. Given.
2. $\measuredangle BAD \cong \measuredangle DAC$	2. Definition of an angle bisector.
3. $AB \cdot AC = AD \cdot AE$	3. Given.
4. $\frac{AB}{AE} = \frac{AD}{AC}$	4. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
5. $\triangle ABD \sim \triangle AEC$	5. S.A.S. Similarity Theorem.
6. $\measuredangle B \cong \measuredangle AEC$	6. Corresponding angles of similar triangles are congruent.

• PROBLEM 6-8

In the accompanying figure, $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$. Segment $\overline{AF}$ is the altitude on $\overline{BC}$. From a point on $\overline{AB}$, call it D, a perpendicular is drawn which is extended to meet $\overleftrightarrow{BC}$. It meets $\overline{BC}$ at point P. Prove that $FC : DB = AC : PB$.



Solution: $\overline{FC}$ and $\overline{AC}$ are sides of $\triangle FCA$ and correspond to sides $\overline{DB}$ and $\overline{PB}$ of $\triangle DBP$. By proving $\triangle FCA \sim \triangle DBP$, we can conclude that $FC : DB = AC : PB$ is a proportion, because there is a theorem which states that corresponding sides of similar triangles are proportional.

The corresponding angles of ΔFCA and ΔDBP have been numbered, in the diagram, by a "prime and no prime" system.

Similarity will be proved by showing two pairs of corresponding angles congruent.

STATEMENT	REASON
1. $\overline{AB} = \overline{AC}$	1. Given.
2. $\sphericalangle 3 \cong \sphericalangle 3'$	2. If two sides of a trignale are congruent, then the angles opposite these sides are congruent.
3. $\overline{AF} \perp \overline{BC}$	3. An altitude drawn to a side is perpendicular to that side.
4. $\overline{PD} \perp \overline{AB}$	4. Given.
5. $\sphericalangle 2$ and $\sphericalangle 2'$ are right angles	5. Perpendicular lines intersect forming right angles.
6. $\sphericalangle 2 \cong \sphericalangle 2'$	6. All right angles are congruent.
7. $\Delta FCA \sim \Delta DBP$	7. A.A. $\cong$ A.A.
8. $\frac{FC\ (\text{opp.} \sphericalangle 1)}{DB\ (\text{opp.} \sphericalangle 1')} = \frac{AC\ (\text{opp.} \sphericalangle 2)}{PB\ (\text{opp.} \sphericalangle 2')}$	8. Corresponding sides of similar triangles are proportional.

• PROBLEM 6-9

Prove that any two regular polygons with the same number of sides are similar.

Solution: For any two polygons to be similar, their corresponding angles must be congruent and their corresponding sides proportional. It is necessary to show that these conditions always exist between regular polygons with the same number of sides.

Let us examine the corresponding angles first. For a regular polygon with n sides, the measure of each central angle is $\frac{360}{n}$ and each vertex angle is $\frac{(n-2)\,180}{n}$. Therefore, two regular polygons with the same number of sides will have corresponding central angles and vertex angles that are all of the same measure and, hence, are all congruent. This fulfills our first condition for similarity.

We must now determine whether or not the corresponding sides are proportional. It will suffice to show that the ratios of the lengths of every pair of corresponding sides are the same.

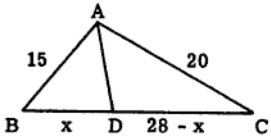
Since the polygons are regular, the sides of each one will be equal. Call the length of the sides of one polygon ℓ_1 and the length of the sides of the other polygon ℓ_2. Hence, the ratio of the lengths of corresponding sides will be ℓ_1/ℓ_2. This will be a constant for any pair of corresponding sides and, hence, the corresponding sides are proportional.

Thus, any two regular polygons with the same number of sides are similar.

PROPORTIONS INVOLVING ANGLE BISECTORS

• PROBLEM 6-10

The sides of a triangle have lengths 15, 20 and 28. Find the lengths of the segment into which the bisector of the angle with the greatest measure divides the opposite side.



Solution: An angle bisector of any triangle divides the side of the triangle opposite the angle into segments proportional to the sides adjacent to the angle. Therefore,

(i) $$\frac{BD}{DC} = \frac{AB}{AC} .$$

If we let BD = x, then DC = 28 - x. Substituting in equation (i), we have

(ii) $$\frac{x}{28 - x} = \frac{15}{20} .$$

Reducing the fraction on the right to lowest terms,

(iii) $$\frac{x}{28 - x} = \frac{3}{4}$$

For real numbers a, b, c, d, it must be true that if

$\frac{a}{b} = \frac{c}{d}$, then ad = cb. Therefore, from equation (iii), we have

(iv) $4x = 3(28 - x).$

Multiplying out, we have

(v) $4x = 84 - 3x$

Adding 3x to both sides,

(vi) $4x + 3x = 84 - 3x + 3x$

(vii) $7x = 84$

Dividing by 7 leaves

(viii) $x = 12.$

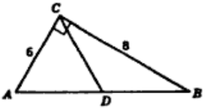
Then,

(ix) $BD = 12$

(x) $DC = 28 - 12 = 16.$

• PROBLEM 6-11

A right triangle has legs of length 6 and 8 inches. $\overline{CD}$ bisects the right angle. Find the lengths of $\overline{AD}$ and $\overline{DB}$.



Solution: We must somehow relate the unknown segment lengths to the given data to derive the required results.

Recall that the bisector of one angle of a triangle divides the opposite side so that the lengths of its segments are proportional to the lengths of the adjacent sides. Thus,

$$\frac{6}{8} = \frac{AD}{DB} .$$

We see that $\frac{8}{8} = \frac{DB}{DB}$ is always true. Hence, adding this to the above proportion, we obtain

$$\frac{6 + 8}{8} = \frac{AD + DB}{DB} .$$

But, AD + DB = AB, the hypotenuse of a right triangle. Applying the Pythagorean theorem, $a^2 + b^2 = c^2$, where a = 6, b = 8, we obtain

$$6^2 + 8^2 = c^2$$

$$36 + 64 = c^2$$

$$100 = c^2.$$

Thus, c = 10, or hypotenuse AB = 10. Substituting we obtain

$$\frac{6 + 8}{8} = \frac{AD + DB}{DB}$$

$$\frac{14}{8} = \frac{AB}{DB}$$

$$\frac{14}{8} = \frac{10}{DB} \quad .$$

Since, in a proportion, the product of the means is equal to the product of the extremes, we obtain

$$14 \cdot DB = 8 \cdot 10,$$

$$DB = \frac{80}{14} = \frac{40}{7} \, .$$

To find AD, we notice that

$$AD = AB - DB$$

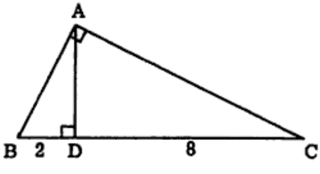
$$= 10 - \frac{40}{7} = \frac{30}{7} \, .$$

Therefore, $AD = \frac{30}{7}$ and $DB = \frac{40}{7}$.

PROPORTIONS INVOLVING RIGHT TRIANGLES

• PROBLEM 6-12

In the figure below, ΔABC is a right triangle. $\overline{AD}$ is an altitude on the hypotenuse $\overline{BC}$. $\overline{BD}$ = 2, DC = 8. Find AD, AB, and AC.



Solution: The geometric mean of two numbers x and z, is defined as that number, y, such that

$$\frac{x}{y} = \frac{y}{z} \ .$$

In the right triangle, the altitude to the hypotenuse is the geometric mean between the segments of the hypotenuse. Therefore,

(i) $$\frac{BD}{AD} = \frac{AD}{DC}$$

Substituting in the values, BD = 2 and DC = 8, the equation becomes

(ii) $$\frac{2}{AD} = \frac{AD}{8} \ .$$

For the proportionality to be true, the product of the means must equal the product of the extremes.

The equation then becomes

(iii) $$(AD)^2 = 2 \cdot 8 = 16$$

(iv) $$AD = \sqrt{16} = 4.$$

To find AB, we note that AB is the hypotenuse of right triangle ΔABD. Since we know the lengths of both legs, we can use the Pythagorean theorem,

(v) $$c^2 = a^2 + b^2.$$

Substituting, we obtain

(vi) $$(AB)^2 = (BD)^2 + (AD)^2$$

(vii) $$(AB)^2 = 2^2 + 4^2 = 4 + 16 = 20$$

(viii) $$AB = \sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2 \cdot \sqrt{5}.$$

Another method of solving for AB is to realize that there is another geometric mean implicit in the triangle. The altitude to the leg divides the hypotenuse so that either leg is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to that leg. Therefore, it must be that

(ix) $$\frac{BD}{AB} = \frac{AB}{BC}$$

We substitute in the values BD = 2 and BC = 10,

(x) $$\frac{2}{AB} = \frac{AB}{10} \ ,$$

(xi) $$(AB)^2 = 2 \cdot 10 = 20,$$

(xii) $$AB = \sqrt{20} \text{ or } 2\sqrt{5}.$$

We find AC in a similar manner. AC is the geometric mean of DC and BC. Therefore,

(xiii) $$\frac{DC}{AC} = \frac{AC}{BC} ,$$

(xiv) $$\frac{8}{AC} = \frac{AC}{10} .$$

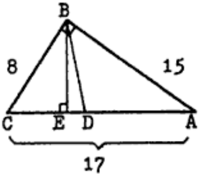
Remember: the product of the means must equal the product of the extremes,

(xv) $$(AC)^2 = 8 \cdot 10 = 80$$

(xvi) $$AC = \sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5}.$$

• PROBLEM 6-13

An altitude and an angle bisector are drawn to the hypotenuse of a right triangle whose sides are 8, 15, and 17 inches long. Find the length of the segment joining the points where the altitude and angle bisector intersect the hypotenuse (see figure). Express your answer to the nearest hundredth.



Solution: In the accompanying figure, ΔABC is a right triangle with AB = 15, BC = 8, and AC = 17. $\overline{BD}$ is the bisector of the right angle and $\overline{BE}$ is the altitude drawn to the hypotenuse. We wish to find DE. With the given information, we can determine (1) DC and (2) EC. Since DE = DC - EC, we can then solve for DE.

(1) To solve for DC, note that the angle bisector divides the opposite side into segments proportional to the sides.

(i) $$\frac{AD}{DC} = \frac{AB}{BC}$$

Note AB = 15 and BC = 8. Since AD + DC = AC = 17, AD can be expressed as 17 - DC. Substituting these results, we have

(ii) $$\frac{17 - DC}{DC} = \frac{15}{8} .$$

We have one equation and one unknown. To solve for DC, multiply both sides by 8 · DC:

(iii) $$8\,(17 - DC) = 15 \cdot DC$$

(iv) $$136 - 8DC = 15DC$$

Adding 8DC to both sides and dividing by 23, we obtain

(v) $$DC = \frac{136}{23}\,.$$

(2) To solve for EC, note that $\overline{BE}$ is an altitude drawn to the hypotenuse of a right triangle and cuts segments $\overline{AE}$ and $\overline{EC}$ on the hypotenuse. Since the length of each leg of the given triangle is the mean proportional between the hypotenuse length and the adjacent segment, we have

(vi) $$BC^2 = EC \cdot AC.$$

We substitute BC = 8, AC = 17, and solve for EC:

(vii) $$8^2 = EC \cdot 17$$

(viii) $$EC = \frac{64}{17}$$

Having found values for DC and EC, we can now solve for DE:

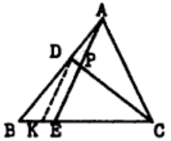
(ix) $$DE = DC - EC$$

(x) $$DE = \frac{136}{23} - \frac{64}{17} = \frac{17(136) - 64(23)}{23(17)}$$

(xi) $$DE = \frac{840}{391} \simeq 2.15.$$

• PROBLEM 6-14

D and E are respective points of side $\overline{AB}$ and $\overline{BC}$ of ΔABC, so that $\frac{AD}{DB} = \frac{2}{3}$ and $\frac{BE}{EC} = \frac{1}{4}$. If $\overline{AE}$ and $\overline{DC}$ meet at P, find $\frac{PC}{DP}$.



Solution: We are asked to find ratio $\frac{PC}{DP}$. To find the value of a ratio $\frac{a}{b}$, we can either (1) find exact values of a and b; (2) express the values of a and b in terms of a common length; or (3) form a proportion with $\frac{a}{b}$ as one side and a known ratio on the other side. Here, no values are given. Thus, PC and DP cannot be solved exactly. Neither is there a common length in which DP and PC can be expressed. The third method calls for a proportion. Proportions imply (1) similar triangles; (2) a triangle cut by a line parallel to one side; and (3) sets of parallel lines. To create proportional sides, draw segment $\overline{DK}$ parallel to $\overline{AE}$ such that K is between B and C. Note that in $\triangle CDK$, line $\overline{PE}$ is parallel to side DK. Therefore, $\frac{PC}{DP} = \frac{EC}{EK}$. Our proportional is formed. $\frac{PC}{DP}$ is on one side. The term on the other side, $\frac{EC}{EK}$, can be found since both EC and EK can be found in terms of BC.

To find EK in terms of BC, we show, (1) in $\triangle BAE$, $\overleftrightarrow{DK} \parallel$ side $\overline{AE}$. Therefore, sides $\overline{BA}$ and $\overline{BE}$ are cut proportionally.

$\frac{KE}{KB} = \frac{AD}{DB}$. Since $\frac{AD}{DB} = \frac{2}{3}$, $\frac{KE}{KB} = \frac{2}{3}$.

(2) From $\frac{KE}{KB}$, we can find KE in terms of BE. $\frac{KE}{KB} = \frac{2}{3}$. Then, $KE = \frac{2}{3}\, KB$, or $KB = \frac{3}{2}\, KE$. Since $BE = KE + BK$, $BE = KE + \frac{3}{2}\, KE = \frac{5}{2}\, KE$. Thus,

$$KE = \frac{2}{5}\, BE.$$

(3) We can find BE in terms of BC. From the given $\frac{BE}{EC} = \frac{1}{4}$ or $BE = \frac{1}{4}\, EC$ or $EC = 4BE$. Since $BC = BE + EC = BE + 4BE = 5\, BE$, then

$$BE = \frac{1}{5}\, BC.$$

(4) We can combine the results of 2 and 3 to find KE in terms of BC. $KE = \frac{2}{5}\, BE$ and $BE = \frac{1}{5}\, BC$. Therefore, $KE = \frac{2}{5}\left(\frac{1}{5}\, BC\right) = \frac{2}{25}\, BC$.

To find EC in terms of BC, note that $\frac{BE}{EC} = \frac{1}{4}$ or $BE = \frac{1}{4}\, EC$. Since $BC = BE + EC = \frac{1}{4}\, EC + EC = \frac{5}{4}\, EC$, then $EC = \frac{4}{5}\, BC$.

Combining these two results, we have

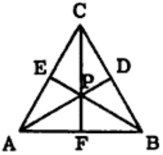
$$\frac{EC}{KE} = \frac{\frac{4}{5}\,BC}{\frac{2}{25}\,BC} = 10.$$

Earlier, we found $\frac{PC}{DP} = \frac{EC}{KE}$. Therefore, $\frac{PC}{DP} = 10$.

PROPORTIONS INVOLVING MEDIANS

• PROBLEM 6-15

In triangle ABC, medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at P, as seen in the figure. If AD = 24 in., find the length of $\overline{AP}$.



Solution: A theorem tells us that the medians of a triangle are concurrent at a point whose distance from any vertex is two-thirds the distance from that vertex to the midpoint of the opposite side. The vertex in question is A, the point of concurrency is P, and the distance to the midpoint of the side opposite A is given by AD. Therefore, the length of $\overline{AP}$ equals $\frac{2}{3}$ the length of $\overline{AD}$. Algebraically, this means

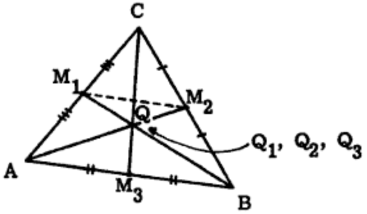
$$AP = \tfrac{2}{3}\,AD$$

$$AP = \tfrac{2}{3}(24'') = 16'' .$$

Therefore, the length of $\overline{AP}$ is 16".

• PROBLEM 6-16

Show that the medians of a triangle are concurrent at a point on each median located two-thirds of the way from each vertex to the opposite side.



Solution: Referring to the figure above, we must show (1) Q is a point on $\overline{AM_2}$, $\overline{BM_1}$, and $\overline{CM_3}$; and (2) that $AQ = \frac{2}{3}AM_2$, $BQ = \frac{2}{3}BM_1$, and $CQ = \frac{2}{3}CM_3$. Since $AM_2 = AQ + QM_2$, $QM_2 = AM_2 - AQ = AM_2 - \frac{2}{3}AM_2 = \frac{1}{3}AM_2$, or $QM_2 = \frac{1}{3}AM_2$. Thus, to prove the second part, we need to show that the point of concurrency divides the median AM_2 into two segments such that the segment near the vertex is twice the length of the segment near the side. In algebraic notation, what we must show for the second part is that $\frac{AQ}{QM_2} = \frac{BQ}{QM_1} = \frac{CQ}{QM_3} = 2$.

Wherever there are proportions, it is wise to look for similar triangles. Suppose we first wish to show that $\frac{AQ}{QM_2} = \frac{2}{1}$. Then we find two triangles such that $\overline{AQ}$ and $\overline{QM_2}$ are corresponding sides. ΔAQB and ΔM_2QM_1 are such triangles. We show similarity by the A-A Similarity Theorem. Then $\frac{AQ}{QM_2} = \frac{BQ}{QM_1} = \frac{AB}{M_1M_2}$. $\frac{AB}{M_1M_2}$ is known. $\overline{M_1M_2}$ is a midline of ΔABC. Therefore, $AB = 2 \cdot M_1M_2$, $\frac{AB}{M_1M_2} = 2$, and $\frac{AQ}{QM_2} = 2$, proving the second part. We can repeat the procedure for the two other sides.

NOTE: It may seem that we have proven the second part without actually showing that the lines are indeed concurrent, but we really don't need the concurrency to show the two-thirds division. In fact, the exact reverse is true. We show that the intersection of any two medians is a point two-thirds the length of each median from the respective vertex. Since there is only one point that is two-thirds the median length from the vertex, the points of intersection must be the same. Concurrency (part one) follows from the two-thirds division (part two).

Given: $\overline{AM_2}$, $\overline{BM_1}$, and $\overline{CM_3}$ are medians of ΔABC.

Prove: (1) Q is a point on $\overline{AM_2}$, $\overline{BM_1}$, and $\overline{CM_3}$.
(2) $AQ = \frac{2}{3}AM_2$, $BQ = \frac{2}{3}BM_1$, $CQ = \frac{2}{3}CM_3$.

Statements	Reasons
1. $\overline{AM_2}$, $\overline{BM_1}$, and $\overline{CM_3}$ are medians of ΔABC	1. Given.
2. Q_1 is the intersection of $\overline{AM_2}$ and $\overline{BM_1}$.	2. Two non-parallel, non-coincident, lines intersect in a point.
3. $\sphericalangle AQ_1B \cong \sphericalangle M_2Q_1M_1$	3. Opposite angles formed by two intersecting lines are congruent.
4. $\overline{M_1M_2}$ is a midline of ΔABC	4. The segment that connects the midpoints of two sides of a triangle is a midline of the triangle.
5. $\overline{M_1M_2} \parallel \overline{AB}$	5. The midline of a triangle is parallel to the third side.

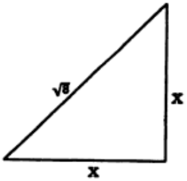
6. $\measuredangle M_1M_2Q \cong \measuredangle QAB$	6. Alternate interior angles of a transversal are congruent.
7. $\Delta M_1QM_2 \sim \Delta AQB$	7. If two angles of one triangle are congruent with the corresponding two angles of a second triangle, then the two triangles are similar.
8. $\frac{M_2Q}{AQ} = \frac{M_1Q}{BQ} = \frac{M_1M_2}{AB}$	8. The corresponding sides of similar triangles are proportional.
9. $AB = 2 \cdot M_1M_2$ or $\frac{M_1M_2}{AB} = \frac{1}{2}$	9. The midline of a triangle is half the length of the third side.
10. $\frac{M_2Q}{AQ} = \frac{M_1Q}{BQ} = \frac{1}{2}$	10. Substitution Postulate.
11. $AM_2 = AQ + M_2Q$ $BM_1 = BQ + M_1Q$	11. Q_1 is the intersection of two segments AM_2 and BM_1. Therefore Q_1 must lie between the endpoints of the segment. The equations at left follow from the definition of betweenness.
12. $M_2Q = \frac{1}{2} AQ_1$ $M_1Q_1 = \frac{1}{2} BQ_1$	12. Multiplication Postulate and Step 10.
13. $AM_2 = 1\frac{1}{2} AQ_1$ $BM_1 = 1\frac{1}{2} BQ_1$	13. Substitution of Step 12 into 11.
14. $AQ_1 = \frac{2}{3} AM_2$ $BQ_1 = \frac{2}{3} BM_1$	14. Multiplication Postulate
15. Q_2 is the intersection of $\overline{AM_2}$ and $\overline{CM_3}$	15. Two non-parallel, non-coincident lines intersect in a unique point.
16. $AQ_2 = \frac{2}{3} AM_2$ $CQ_2 = \frac{2}{3} CM_3$	16. Obtained by repeating procedure used from Steps 3 through 13. The midline is now $\overline{M_1M_3}$ not $\overline{M_1M_2}$.
17. Q_3 is the intersection of medians $\overline{CM_3}$ and $\overline{BM_1}$	17. Two non-parallel non-coincident lines intersect in a unique point.
18. $CQ_3 = \frac{2}{3} CM_3$ $BQ_3 = \frac{2}{3} BM_1$	18. Obtained by repeating procedure used from Steps 3 through 13. The midline is now $\overline{M_1M_3}$.
19. Q_1 is a point on segment $\overline{AM_2}$ such that $AQ_1 = \frac{2}{3} AM_2$. Q_2 is a point on segment $\overline{AM_2}$ such that $AQ_2 = \frac{2}{3} AM_2$.	19. Repetition of results from Steps 14 and 16.
20. Point Q_1 is point Q_2	20. On a given segment, there is only one point on the segment that is a given distance from a given endpoint.

21. Q_2 is a point on segment $\overline{CM_3}$ such that $CQ_2 = \frac{2}{3} CM_3$. Q_3 is a point on segment $\overline{CM_3}$ such that $CQ_3 = \frac{2}{3} CM_3$.	21. Repetition of results from Steps 16 and 18.
22. Point Q_2 is point Q_3	22. Same reason as Step 20.
23. Let point $Q = Q_1 = Q_2 = Q_3$	23. From Steps 20 and 22, we have shown $Q_1 = Q_2 = Q_3$ and here we give this common point of intersection of the three medians a more general name.
24. Q is a point on $\overline{AM_2}$, $\overline{BM_1}$, and $\overline{CM_3}$	24. Follows from Step 23 and the definitions of C_1, C_2, and C_3 .
25. $AQ = \frac{2}{3} AM_2$ $BQ = \frac{2}{3} BM_1$ $CQ = \frac{2}{3} CM_3$	25. Substitution Postulate (Substitute Q for Q_1, Q_2, and Q_3 in Steps 14, 16, and 18.)

PYTHAGOREAN THEOREM AND APPLICATIONS

• PROBLEM 6-17

The legs of a certain right triangle are equal and the hypotenuse is $\sqrt{8}$. What is the length of either leg of the triangle?



Solution: Recall the Pythagorean Theorem, $a^2 + b^2 = c^2$, where a and b are the lengths of the two legs of a right triangle, and c is the length of the hypotenuse. We can use this to solve for the length of the legs. Let x = the length of each of the equal legs. Substituting in the above formula, $x = a = b$ and $\sqrt{8}$ for c,

$$x^2 + x^2 = (\sqrt{8})^2$$

$$2x^2 = 8$$

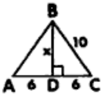
$$x^2 = 4$$

$$x = 2$$

Therefore, each leg is 2 units in lengths.

• PROBLEM 6-18

In an isosceles triangle, the length of each of the congruent sides is 10 and the length of the base is 12. Find the length of the altitude drawn to the base.



Solution: In the figure shown, altitude $\overline{BD}$ has been drawn. By a theorem, we know that $\overline{BD} \perp \overline{AC}$ and that $\overline{BD}$ bisects $\overline{AC}$. Therefore, $\overline{AD} \cong \overline{DC}$ and they both measure 6.

Since $\overline{BD} \perp \overline{AC}$, we can conclude that ΔBDC is a right triangle and then apply the Pythagorean Theorem. As applied here, it states that

$$(DC)^2 + (BD)^2 = (BC)^2.$$

We were given that BC = 10, and we have found that DC = 6. If we let x = BD and substitute into the equation above, we arrive at

$$6^2 + x^2 = 10^2$$

$$36 + x^2 = 100$$

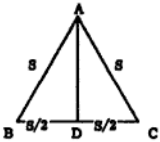
$$x^2 = 64$$

$$x = 8.$$

Therefore, the length of the altitude drawn to the base is 8.

• PROBLEM 6-19

Let ΔABC be an equilateral triangle of side length s. Let D be the midpoint of $\overline{BC}$. Compute AD. (See figure.)



Solution: We will apply the Pythagorean Theorem to ΔABD to obtain AD.

$$(AB)^2 = (AD)^2 + (BD)^2$$

Hence,
$$(AD)^2 = (AB)^2 - (BD)^2$$

$$AD = \sqrt{(AB)^2 - (BD)^2}$$

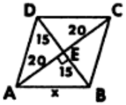
Since BD = DC = s/2, and AB = s, therefore,

$$AD = \sqrt{s^2 - s^2/4}$$

$$AD = \sqrt{3s^2/4} = s\,\frac{\sqrt{3}}{2}$$

• PROBLEM 6-20

The lengths of the diagonals of the rhombus shown in the figure are 30 and 40. Find the perimeter of the rhombus.



Solution: Since the diagonals of a rhombus are perpendicular bisectors of each other, ΔAEB is a right triangle in which $EB = \frac{1}{2}(30)$, or 15, and $AE = \frac{1}{2}(40)$, or 20. With this in mind, we can calculate the length of the hypotenuse $\overline{AB}$, of ΔAEB by using the Pythagorean Theorem:

$$(AB)^2 = (EB)^2 + (AE)^2.$$

$\overline{AB}$ is one side of a rhombus. All sides of a rhombus are congruent, thus, the perimeter can be calculated by multiplying AB by 4. If we let AB = x, by substitution,

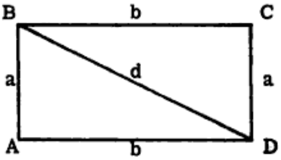
$$x^2 = (15)^2 + (20)^2 = 225 + 400$$

or
$$x^2 = 625; \quad \text{solving } x = 25.$$

Then, the perimeter = 4x or 4 (25) = 100.

• PROBLEM 6-21

Suppose that the area of a rectangle is 300 sq. in., and that the length of its diagonal is 25 in. Find the lengths of the sides of the rectangle.



Solution: Let the sides of the rectangle be labeled as in the figure. Then the area of the rectangle A(R) is

$$A(R) = ba = 300 \text{ sq. in.} \tag{1}$$

Since ΔABD is a right triangle,

$$d^2 = a^2 + b^2 \tag{2}$$

by Pythagoras' Theorem. Noting that d = 25 in., we can write:

$$625 \text{ sq. in.} = a^2 + b^2 \tag{3}$$

We now have 2 equations ((1) and (2)) and 2 unknowns (a and b). We may, therefore, solve explicitly for a and b. First solve (1) for a:

$$a = \frac{300 \text{ sq. in.}}{b} \tag{4}$$

Inserting (4) into (3),

$$625 \text{ sq. in.} = \left(\frac{300 \text{ sq. in.}}{b}\right)^2 + b^2$$

$$625 \text{ sq. in.} = \frac{90000 \text{ in}^4}{b^2} + \frac{b^4}{b^2}$$

$$625 \text{ sq. in.} = \frac{b^4 + 90000 \text{ in}^4}{b^2}$$

$$b^4 - (625 \text{ sq. in.})\, b^2 + 90000 \text{ in}^4 = 0$$

Let $$x = b^2 \tag{5}$$

then, $$x^2 - (625 \text{ sq. in.})\, x + 90000 \text{ in}^4 = 0 \tag{6}$$

We solve this by means of the quadratic formula. If an equation is of the form

$$ax^2 + bx + c = 0,$$

its roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

In equation (6), $a = 1$, $b = -625$ sq. in., and $c = 90{,}000\ \text{in}^4$. Using these in (7) yields

$$x = \frac{625\ \text{sq. in.} \pm \sqrt{(-625\ \text{sq. in.})^2 - (4)(1)(90{,}000\ \text{in}^4)}}{(2)(1)}$$

$$x = \frac{625\ \text{sq. in.} \pm \sqrt{30625\ \text{in}^4}}{2}$$

$$x = \frac{625\ \text{sq. in.} \pm 175\ \text{sq. in.}}{2}$$

$x = 400$ sq. in. or 225 sq. in.

But, from (5), $x = b^2$; hence,

$b^2 = 400$ sq. in. or 225 sq. in.

Therefore, $b = 20$ in. or 15 in. (8)

(Negative roots are eliminated because b is a distance and must be positive.)

Using (8) in (4), we find

$a = \frac{300}{20}$ in., or $\frac{300}{15}$ in.

$a = 15$ in., or 20 in.

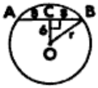
Hence, the lengths are

$a = 15$ in., $b = 20$ in.

or, $a = 20$ in., $b = 15$ in.

• PROBLEM 6-22

A chord, 16 inches long, is 6 inches from the center of a circle. Find the length of the radius of the circle.



Solution: The distance of a chord from the center of a circle is measured on a line perpendicular to the chord from the center of the circle. Therefore, as labeled in the accompanying figure, $\overline{OC} \perp \overline{AB}$ and ΔOCB is a right triangle.

In addition to being perpendicular to $\overline{AB}$, the segment $\overline{OC}$ also intersects chord $\overline{AB}$ at its midpoint. Hence, $\overline{OC}$ bisects $\overline{AB}$. Therefore, $\overline{AC} \cong \overline{BC}$, and they both measure 8 in. We are given that OC = 6 in.

Since ΔOCB is a right triangle, we can apply the Pythagorean Theorem to calculate the length of the hypotenuse of the triangle, which is also the radius of the circle.

In this problem,

$$(OB)^2 = (OC)^2 + (CB)^2.$$

If we let r = the length of radius $\overline{OB}$, by substitution

$$r^2 = 6^2 + 8^2$$

$$r^2 = 36 + 64$$

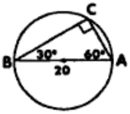
$$r^2 = 100$$

$$r = 10.$$

Therefore, the length of the radius of the circle is 10 inches.

• PROBLEM 6-23

In a circle, angle ABC, formed by diamater $\overline{AB}$ end chord $\overline{BC}$, is 30°. If the length of the diameter of the circle is 20, find the length of chord $\overline{AC}$ and, in radical form, the length of chord $\overline{BC}$.



Solution: From the figure shown, we see that $\sphericalangle$ BCA is inscribed in a semi-circle, therefore is 90°. Angle CAB is the third angle of ΔABC, and it measures 180° - (m $\sphericalangle$ ABC + m $\sphericalangle$ BCA), or 180° - (30° + 90°) = 60°. Therefore, ΔABC is a 30-60-90 triangle.

Chord $\overline{AC}$ is opposite the 30° angle, and, accordingly, measures one-half of the length of hypotenuse $\overline{AB}$. Therefore,

$$\overline{AC} = \tfrac{1}{2}\,\overline{AB} = \tfrac{1}{2}(20) = 10.$$

Chord $\overline{BC}$ is opposite the 60° angle, and its length is given by ½ the length of the hypotenuse times $\sqrt{3}$. Thus,

$$\overline{BC} = \tfrac{1}{2}\ \overline{AB}\ \sqrt{3} = \tfrac{1}{2}\ (20)\ \sqrt{3} = 10\ \sqrt{3}.$$

Therefore, the length of $\overline{AC}$ is 10 and the length of $\overline{BC}$ is 10 $\sqrt{3}$.

• PROBLEM 6-24

Which of the following are Pythagorean Triples?
(a) 1, 2, 3; (b) 3, 4, 5 ; (c) 5, 6, 7; (d) 5, 12, 13;
(e) 11, 60, 61; (f) 84, 187, 205.

Solution: A set of 3 positive integers, a, b, and c, (where c is the largest of the set) is a Pythagorean Triple if and only if

$$c^2 = a^2 + b^2$$

(i.e. they satisfy Pythagorean Theorem).

(a) For (1, 2, 3) to be a Pythagorean triple, it must be true that

$$3^2 = 2^2 + 1^2 \qquad \text{or} \qquad 9 = 4 + 1 = 5.$$

Since $9 \neq 5$, therefore, (1, 2, 3) is not a Pythagorean Triple.

(b) $c = 5,\ a = 3,\ b = 4.$

$$c^2 \overset{?}{=} a^2 + b^2$$

$$25 \overset{?}{=} 9 + 16$$

Since $25 = 25,$

(3, 4, 5) is a Pythagorean Triple.

(c) $c = 7,\ a = 5,\ b = 6$

$$c^2 \overset{?}{=} a^2 + b^2$$

$$49 \overset{?}{=} 25 + 36$$

$$49 \neq 61$$

Therefore, (5, 6, 7) is not a Pythagorean Triple.

(d) $c = 13,\ a = 5,\ b = 12.$

$$c^2 \overset{?}{=} a^2 + b^2$$

$$169 \overset{?}{=} 25 + 144$$

$$169 = 169$$

Hence, (5, 12, 13) is a Pythagorean Triple.

(e) $c = 61$, $a = 11$, $b = 60$

$$c^2 \overset{?}{=} a^2 + b^2$$

$$3721 \overset{?}{=} 121 + 3600$$

$$3721 = 3721$$

Therefore, (11, 60, 61) is a Pythagorean Triple.

(f) $c = 205$, $a = 84$, $b = 187$

$$c^2 \overset{?}{=} a^2 + b^2$$

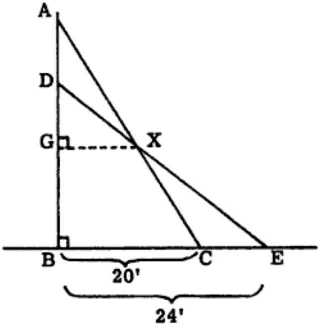
$$42025 \overset{?}{=} 7056 + 34969$$

$$42025 = 42025$$

Therefore, (84, 187, 205) is a Pythagorean Triple.

• PROBLEM 6-25

A 25-foot ladder leans against the wall such that the base of the ladder is 20 feet from the wall. The ladder slides until the base is 24 feet from the wall. Locate the point X that is common to both ladder positions (see diagram).



Solution: $\overline{AC}$ is the original position of the ladder; $\overline{DE}$, the final position. The length of the ladder is constant. Hence, $AC = DE = 25$ ft. The original distance from the base of the ladder to the wall, $BC = 20$; the final distance $BE = 24$. To find the height of the ladder top in both cases, by using the Pythagorean Theorem:

The original height

$$AB = \sqrt{(AC)^2 - (BC)^2} = \sqrt{25^2 - 20^2} = \sqrt{225} = 15 \text{ ft.}$$

The final height

$$DB = \sqrt{(DE)^2 - (BE)^2} = \sqrt{25^2 - 24^2} = \sqrt{49} = 7 \text{ ft.}$$

We wish to find the point X common to $\overline{DE}$ and $\overline{AC}$.

We draw the line $\overline{GX} \perp \overline{AB}$. Remember that a line drawn through two sides of a triangle and parallel to the third side forms 2 similar triangles. Thus, $\triangle AGX \sim \triangle ABC$ and also $\triangle DGX \sim \triangle DBE$. We can relate these two similarities to find GX and AG. By the Pythagorean Theorem, we can find AX. Thus, we will have the location of X: a distance AX from the top of the ladder in the original position.

$\triangle AGX \sim \triangle ABC$ by the A.A. Similarity Theorem ($\sphericalangle A \cong \sphericalangle A$, and $\sphericalangle AGX \cong \sphericalangle ABC$ = right angles). Similarly, $\triangle DGX \sim \triangle DBE$, since $\sphericalangle D \cong \sphericalangle D$ and $\sphericalangle AGX \cong \sphericalangle ABC$. Since corresponding sides of similar triangles are proportional, we have

(i) $$\frac{AG}{GX} = \frac{AB}{BC} = \frac{15}{20} = \frac{3}{4}$$

(ii) $$\frac{DG}{GX} = \frac{DB}{BE} = \frac{7}{24}.$$

We can relate AG and DG by expressing AG in terms of GX and GX in terms of DG. From (i) and (ii), we obtain

(iii) $$AG = \frac{3}{4}\,GX$$

(iv) $$DG = \frac{7}{24}\,GX \qquad \text{or} \qquad GX = \frac{24}{7}\,DG.$$

By substitution, we have,

(v) $$AG = \frac{3}{4}\,GX = \frac{3}{4}\left(\frac{24}{7}\,DG\right) = \frac{18}{7}\,DG.$$

To solve for AG and DG, we need only one more relation. Note that AG = AD + DG. AD is the difference in ladder heights, AB - DB = 15 - 7 = 8. Then, by substitution,

(vi) $$AG = AD + DG = 8 + DG, \quad \text{or} \quad DG = AG - 8.$$

Combining (v) and (vi), we obtain:

(vii) $$AG = \frac{18}{7}(AG - 8) = \frac{18}{7}\,AG - \frac{144}{7}$$

(viii) $$\frac{18}{7}\,AG - AG = \frac{144}{7}$$

(ix) $$\frac{11}{7}AG = \frac{144}{7} \quad \text{or} \quad AG = \left(\frac{7}{11}\right)\left(\frac{144}{7}\right) = \frac{144}{11}.$$

Thus, $AG = \frac{144}{11}$. To find GX, remember that by (iii), $AG = \frac{3}{4}GX$. Then, $GX = \frac{4}{3}\left(\frac{144}{11}\right) = \frac{192}{11}$.

We could stop here and say that point X, in terms of the original ladder position, is $\frac{144}{11}$ ft. down from the top of the wall and $\frac{192}{11}$ ft. from the wall; but it may be easier to measure distances along the ladder. Therefore, we can find AX. Note that AX is the hypotenuse of right ΔAGX. By the Pythagorean Theorem,

$$(AX)^2 = (AG)^2 + (GX)^2 = \left(\frac{144}{11}\right)^2 + \left(\frac{192}{11}\right)^2.$$

DIGRESSION: Whenever cumbersome calculations present themselves, it is best to look for easier methods. We are asked to square two larger numbers and then to find a square root. We wonder, therefore, whether or not we are squaring unnecessary factors. Note that

$$\frac{144}{11} = \frac{48}{11}(3) \quad \text{and} \quad \frac{192}{11} = \frac{48}{11}(4).$$

Then, $$(AX)^2 = \left(\frac{48}{11}\times 3\right)^2 + \left(\frac{48}{11}\times 4\right)^2$$

$$= \left(\frac{48}{11}\right)^2 \times 3^2 + \left(\frac{48}{11}\right)^2 \times 4^2 = \left(\frac{48}{11}\right)^2 (3^2 + 4^2)$$

$$= \left(\frac{48}{11}\right)^2 (9 + 16) = \left(\frac{48}{11}\right)^2 (25).$$

Then, $$AX = \sqrt{\left(\frac{48}{11}\right)^2 (25)} = \sqrt{\left(\frac{48}{11}\right)^2}\sqrt{25} = \frac{48}{11}(5) = \frac{240}{11}$$

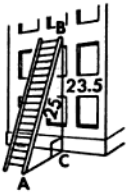
Therefore, the point X is $\frac{240}{11}$ feet down from the top end of the ladder.

Thus, the point in space that is occupied by the ladder in the original position and the final position is $\frac{240}{11}$ or 21.8 ft. from the top of the ladder along the original line of the ladder.

TRIGNOMETRIC RATIOS

• PROBLEM 6-26

A ladder 25 feet long leans against a building and reaches a point 23.5 feet above the ground. Find, to the nearest degree, the angle which the ladder makes with the ground.



Solution: As seen in the accompanying drawing, the building forms a right triangle with the ground. Angle A is the angle the question asks us to determine. We are given the length of the side opposite ∢ A, 23.5, and the length of the hypotenuse (the ladder) 25. With this information, we can calculate the measure of the angle by using the sine ratio.

$$\text{Sin A} = \frac{\text{length of leg opposite} \sphericalangle \text{A}}{\text{length of hypotenuse}} = \frac{BC}{AB}\ .$$

By substitution, $\text{Sin A} = \frac{23.5}{25} = 0.9400$.

Now, we consult a standard sine table to determine which angles have sines near 0.9400.

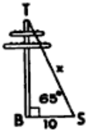
We find that Sin 70° = 0.9397 and Sin 71° = 0.9455. Since 0.9400 is closer to Sin 70°, we can conclude that m ∢ A is closer to 70°.

Therefore, the ladder makes a 70° angle with the ground.

• PROBLEM 6-27

A guy wire reaches from the top of a pole to a stake in the ground. The stake is 10 feet from the foot of the pole. The wire makes an angle of 65° with the ground. Find, to the nearest foot, the length of the wire.

Solution: A right triangle is formed by the pole, the wire and the ground. The wire acts as the hypotenuse of this triangle, and its length is the unknown to be determined. An acute angle and the length of the side adjacent to it are given and, to find the length of the hypotenuse, the cosine ratio can be applied.



$$\text{Cos } S = \frac{\text{length of leg adjacent to } \sphericalangle S}{\text{length of hypotenuse}} = \frac{SB}{ST}\,.$$

Let $x = ST$ and, by substitution, $\text{Cos } 65° = \frac{10}{x}$.

In a standard cosine table, we find Cos 65° = 0.4226.

Substitute this value into the equation

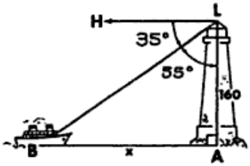
$$0.4226 = \frac{10}{x}$$

$$x = \frac{10}{0.4226} = 23.6.$$

Therefore, the guy wire is 24 feet long, correct to the nearest foot.

• PROBLEM 6-28

From the top of a lighthouse 160 ft. above sea level, the angle of depression of a boat at sea is 35°. Find, to the nearest foot, the distance from the boat to the foot of the lighthouse.



Solution: The distance from the foot of the lighthouse to the boat is represented by BA, (see figure), which is the unknown we wish to find. We will determine BA by an application of the tangent ratio.

The angle of depression, ∢ BLH, is complementary to ∢ BLA. Since m ∢ BLH = 35°, the measure of ∢ BLA is 90° - 35°, or 55°. We cannot use the angle of depression because it is on the exterior of the right triangle.

Given the measure of an acute angle of a right triangle and the length of the side adjacent to it, we can determine the length of the opposite side by a direct application of the tangent ratio.

$$\tan \sphericalangle BLA = \frac{\text{length of leg opposite } \sphericalangle BLA}{\text{length of leg adjacent } \sphericalangle BLA} = \frac{BA}{LA}$$

Let $x = BA$, thus $\tan 55° = \frac{x}{160}$. Consult a standard tangent table to find $\tan 55° = 1.4281$, and substitute this to obtain

$$1.4281 = \frac{x}{160}$$

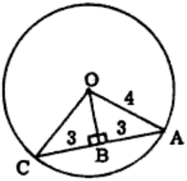
$$x = (1.4281)\, 160$$

$$x = 228.496$$

Therefore, the boat is 228 ft. away from the foot of the lighthouse (to the nearest foot).

• PROBLEM 6-29

In a circle whose radius is 4 in., find, to the nearest inch, the length of the minor arc intercepted by a chord 6 inches in length.



Solution: To find the length of minor arc $\overset{\frown}{AC}$, we must first find the measure of the arc in degrees. Then we will apply the arc length formula which states that

$$\text{arc length} = \frac{\text{degree measure of arc}}{360} \times \text{Circumference}.$$

Circumference $= C = 2\pi r$, where r = radius of circle. By substitution, $C = 2\pi(4) = 8\pi$. $\sphericalangle AOC$ is the central angle of the arc in question, $\overset{\frown}{AC}$. $m\sphericalangle AOC$ is what we wish to determine. Δ AOC is isosceles because, as radii ofcircle O, $\overline{OC} \cong \overline{OA}$. Draw the altitude to $\overline{AC}$. By a theorem, we know that altitude $\overline{OB}$ bisects vertex angle $\sphericalangle AOC$. Therefore, $m\sphericalangle AOC = m\sphericalangle AOB + m\sphericalangle COB$ and, since $\overline{OB}$ is the bisector of $\sphericalangle AOC$, $m\sphericalangle AOB = m\sphericalangle COB = 1/2 m\sphericalangle AOC$.

Hence, $m\sphericalangle AOC = 2(m\sphericalangle AOB)$. ΔAOB is a right triangle, since $OB \perp AC$.

Since we know the hypotenuse and the side opposite $\measuredangle AOB$, we can use the sine ratio to determine the measure of $\measuredangle AOB$.

$$\text{Sin } (\measuredangle AOB) = \frac{\text{leg opposite } \measuredangle AOB}{\text{hypotenuse}} = \frac{AB}{OA}$$

$\overline{OA}$ is a radius given to be 4 in. Since a line drawn from the center of a circle perpendicular to a chord bisects that chord, $AB = 1/2(AC) = 1/2(6) = 3$

Therefore, $\sin \ (\measuredangle AOC) = \frac{3}{4} = 0.75$

The measure of the angle whose sine is closest to 0.75 is 49 .

Hence, by substitution

$$m\measuredangle AOC = 2(m\measuredangle AOB) = 2(49^\circ) = 98^\circ$$

Then, length of minor arc $\overset{\frown}{AC} = \frac{98}{360} \times 8\pi = 2.2\pi$

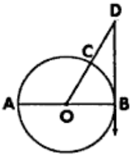
Using $\pi = 3.14$, $2.2\pi = 6.9$

Therefore, to the nearest inch, length of minor arc $\overset{\frown}{AC}$ = 7in.

• PROBLEM 6-30

In circle O, $\overline{AB}$ is a diameter. Radius $\overline{OC}$ is extended to meet tangent $\overline{BD}$ at D. The measures of arc $\overset{\frown}{AC}$ and arc $\overset{\frown}{CB}$ are in the ratio 2:1 and $\overline{AB} = 32$.

(a) Find the number of degrees contained in arc $\overset{\frown}{CB}$.
(b) Find, to the nearest integer, the perimeter of the figure bounded by $\overline{BD}$, $\overline{DC}$ and arc $\overset{\frown}{CB}$. [$\pi = 3.14$] (See figure)



Solution: (a) To find the measure of arc $\overset{\frown}{CB}$, we must make use of two known facts. First, arc $\overset{\frown}{ACB}$ is a semi-circle, and, as such, contains 180°. Second, the composite arcs of $\overset{\frown}{ACB}$; $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$, have measures in a ratio of 2:1. If we let x = the measure of arc $\overset{\frown}{CB}$, then $2x$ = the measure of arc $\overset{\frown}{AC}$. Since the measure of $\overset{\frown}{ACB}$ = measure of $\overset{\frown}{AC}$ + measure of $\overset{\frown}{CB}$, and $\overset{\frown}{ACB}$ is a semicircle, we may write $180^\circ = 2x + x = 3x$ or $x = 60^\circ$. Therefore, the number of degrees contained in arc $\overset{\frown}{CB}$ is 60°.

(b) To find the perimeter of the figure in question, we must determine the lengths of arc $\overset{\frown}{CB}$ and segments

$\overline{DC}$ and $\overline{DB}$, and then sum these quantities. We will first calculate the length of arc $\overset{\frown}{CB}$.

$$\text{length of arc } \overset{\frown}{CB} = \frac{\text{measure of central angle COB}}{360°} \times \text{Circumference}$$

of circle O. Arc $\overset{\frown}{CB}$ has a measure of 60°, and its central angle, ∢COB, by theorem, has the same measure. Circumference of circle O = $2\pi r$, where r is the radius of circle O. We are given diameter AB = 32. Therefore, any radius = $\frac{1}{2}(32)$ or 16.

Circumference of circle O = $2\pi(16) = 32\pi = 32(3.14) = 100.5$.

Therefore, length of arc $\overset{\frown}{CB} = \frac{60°}{360°} \times 100.5 = 16.75$.

Next, we need to find BD and DC. △ODB is a right triangle, since tangent $\overline{BD}$ is perpendicular to radius $\overline{OB}$. (A tangent to a circle is always perpendicular to a radius drawn to the point of tangency.) Hence, the various trigonometric ratios can be employed to find the lengths of the unknown segments. We know m∢DOB = 60°, and the leg adjacent to it, $\overline{OB}$, has a length of 16. We can find the length of the opposite leg, $\overline{DB}$, by using the tangent ratio, and the length of the hypotenuse, $\overline{OD}$, by using the cosine ratio.

$$\tan(\measuredangle DOB) = \frac{\text{length of leg opposite } \measuredangle DOB}{\text{length of leg adjacent } \measuredangle DOB} = \frac{DB}{OB}.$$

By substitution, $\tan 60° = \frac{DB}{16}$. According to a standard tangent table, $\tan 60° = 1.7321$, hence $1.7321 = \frac{DB}{16}$, DB = (16)(1.7321). Therefore, BD = 27.7. The final segment, whose length we need, is DC. $\overline{DC}$ = OD − OC. We know OC = 16, because it is a radius of O. $\overline{OD}$ is the hypotenuse of rt. △ODB. We will use the cosine ratio to find its length.

$$\text{cosine}(\measuredangle DOB) = \frac{\text{length of leg adjacent } \measuredangle COB}{\text{length of hypotenuse}} = \frac{OB}{OD}$$

OB = 16 and m∢DOB = 60°. By substitution, $\cos 60° = \frac{16}{OD}$

$OD = \frac{16}{\cos 60°}$. According to a standard cosine table, $\cos 60° = .5$.

$$OD = \frac{16}{5} = 32$$

Therefore, length of $\overline{DC}$ = OD − OC = 32 − 16 = 16. Hence, the perimeter of the figure bounded by $\overline{BD}$, $\overline{DC}$, and arc $\overset{\frown}{CB}$ is the sum of the measures of these individual parts, or 27.7 + 16 + 16.75 = 60.45. The perimeter of the required figure is 60, to the nearest integer.

INTERIOR AND EXTERIOR ANGLES

• PROBLEM 6-31

If an exterior angle of a polygon is obtuse, what can one say about the corresponding interior angle?

Solution: The exterior angle of a polygon and the corresponding interior angle are supplementary. This means that the sum of the measures of these angles is 180°. If one of the 2 angles is obtuse (i.e. has measure greater than 90°), then the other angle must have a measure less than 90° (i.e., it is acute).

• PROBLEM 6-32

Show that the measure of the exterior angle of a regular n-sided polygon is given by the formula $\frac{360}{n}$.

Solution: An exterior angle of a polygon is defined to be an angle that forms a linear pair (i.e., supplementary) with an interior angle of a polygon. Thus,

(i) $m\measuredangle \text{ exterior} + m\measuredangle \text{ interior} = 180^\circ$.

To find $m\measuredangle$ interior, recall that the sum of the interior angles of an n-sided polygon is $(n-2)\cdot 180^\circ$. Since all interior angles of a regular polygon are congruent, each of the n interior angles has measure $\frac{(n-2)\cdot 180^\circ}{n}$. Substituting $m\measuredangle \text{ interior} = \frac{(n-2)}{n}\cdot 180^\circ$ in equation (i), we obtain

(ii) $m\measuredangle \text{ exterior} + \frac{n-2}{n}\cdot 180^\circ = 180^\circ$.

Solving for $m\measuredangle$ exterior, we have:

(iii)
$$\begin{aligned} m\measuredangle \text{ exterior} &= 180^\circ - \frac{n-2}{n}\cdot 180^\circ \\ &= 180^\circ\left(1 - \frac{n-2}{n}\right) = 180^\circ\left(\frac{2}{n}\right) \\ &= \frac{360^\circ}{n}. \end{aligned}$$

• PROBLEM 6-33

Find the number of degrees contained in each interior angle of a regular hexagon.

Solution: A regular hexagon is a 6 sided polygon in which all sides and all interior angles are of equal measure. A theorem tells us that the measure of each interior angle of a regular polygon of n sides is $\frac{180^\circ(n-2)}{n}$. In this case, n equals 6.

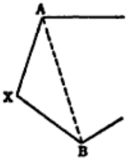
Therefore, each angle's measure is calculated as follows:

$$\frac{180^\circ(n-2)}{n} = \frac{180^\circ(6-2)}{6} = \frac{180^\circ(4)}{6} = 120^\circ.$$

Therefore, each angle of a regular hexagon measures 120°.

• PROBLEM 6-34

$\overline{AX}$ and $\overline{BX}$ are two adjacent sides of a regular polygon. If the measure of angle ABX equals $\frac{1}{3}$ the measure of angle AXB, how many sides has the regular polygon?



Solution: We first determine the measure of vertex $\measuredangle$ AXB. Since we are told the polygon is regular, the number of sides will be uniquely determined. The measure of each vertex angle of a regular polygon is given by the expression $\frac{180(n-2)}{n}$, where n is the number of sides. We will set $m \measuredangle AXB = \frac{180(n-2)}{n}$ and solve for n.

Draw $\overline{AB}$ to complete Δ AXB. Since $\overline{AX}$ and $\overline{BX}$ are sides of a regular polygon, $\overline{AX} \cong \overline{BX}$. Therefore, Δ AXB is isosceles.

The measures of the base angles of an isosceles triangle are equal. As such, $m \measuredangle ABX = m \measuredangle XAB$. We are given that $m \measuredangle AXB = 3\, m \measuredangle ABX$. The measure of the angles of a triangle sum to 180. Therefore, $m \measuredangle AXB + m \measuredangle ABX + m \measuredangle XAB = 180$. By substitution,

$$3m \measuredangle ABX + m \measuredangle ABX + m \measuredangle ABX = 180$$
$$5m \measuredangle ABX = 180$$
$$m \measuredangle ABX = \frac{180}{5} = 36^\circ.$$

Therefore, $m \measuredangle AXB = 3(36) = 108$. Set $108 = \frac{180(n-2)}{n}$ and solve for n.

$$108n = 180n - 360$$
$$-72n = -360$$
$$n = \frac{360}{72} = 5.$$

Therefore, the regular polygon has 5 sides, i.e. it is a pentagon.

• PROBLEM 6-35

For a regular polygon of six sides, find the number of degrees contained in (a) each central angle, (b) each interior angle (c) each exterior angle. (See figure.)

Solution: (a) The central angle of a regular polygon is an angle formed by two radii of the polygon drawn to consecutive vertices of the polygon. Its measure is equal to $\frac{360°}{n}$, where n = the number of sides in the polygon. In the figure shown, $\measuredangle AOB$ is a central angle. Hence, $m\measuredangle AOB = \frac{360°}{6} = 60°$.

Therefore, each central angle contains 60°.

(b) An interior angle of a regular polygon is formed by two consecutive sides of the polygon meeting at a vertex. $\measuredangle ABC$ is such an angle. Its measure is given by $\frac{(n - 2)\ 180°}{n}$, where n = the number of sides. In this example, n = 6 and, therefore, by substitution, each angle measures $\frac{(6 - 2)\ 180°}{6}$, or 120°.

Therefore, each interior angle contains 120°.

(c) An exterior angle of a regular polygon is formed outside the regular polygon by one side and the outward extension of an adjacent side. $\measuredangle CBG$ is an exterior angle. (See figure.) Its measure equals $\frac{360}{n}$ degrees. In this problem $\frac{360°}{n} = \frac{360°}{6} = 60°$.

Therefore, each exterior angle contains 60°.

CHAPTER 7

CIRCLES

CENTRAL ANGLES AND ARCS

• PROBLEM 7-1

In a circle whose radius is 8 inches, find the number of degrees contained in the central angle whose arc length is 2π inches.

Solution: The measure of a central angle is equal to the measure of the arc it intercepts.

The ratio of arc length to circumference, in linear units, will be equal to the ratio of arc length to circumference as measured in degrees.

If n = the number of degrees in the arc 2π inches long,

Then $$\frac{\text{length of arc}}{\text{circumference}} = \frac{n}{360°}$$

By substitution, $$\frac{2\pi \text{in.}}{2\pi(8 \text{ in.})} = \frac{n}{360°}$$

$$\frac{1}{8} = \frac{n}{360°}$$

$$360° = 8n$$

$$n = 45°$$

Therefore, the central angle contains 45°.

• PROBLEM 7-2

Find the number of degrees, to the nearest degree, in an angle subtended at the center of a circle by an arc 5 ft. 10 in. in length. The radius of the circle is 9 ft. 4 in.

Solution: Let us convert the mixed unit measures into single unit measure.

$$(12 \text{ in.} = 1 \text{ ft.})$$

$$5 \text{ ft. } 10 \text{ in.} = 5(12) + 10 = 70 \text{ in.}$$

$$9 \text{ ft. } 4 \text{ in.} = 9(12) + 4 = 112 \text{ in.}$$

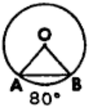
We solve this problem by determining what fraction of the total circumference the arc occupies and then multiplying this by 360° to find the degree measure of the arc. The angle subtended at the center is a central angle and contains the same number of degrees as the arc.

Circumference $C = \pi 2\, r$, where r = the radius length. By substitution, $C = 2\pi(112) = 224\pi$ in. The fraction of the total circumference is given by $\frac{70}{224\pi} = \frac{70}{(224)(3.14)} = \frac{70}{704}$.

Hence, there are $(360)\left(\frac{70}{704}\right)$ or 35° 48' contained in this arc and in its central angle. To the nearest degree, the answer is 36°.

• PROBLEM 7-3

In circle O, the measure of $\widehat{AB}$ is 80° Find the measure of ∢A.



Solution: The accompanying figure shows that $\widehat{AB}$ is intercepted by central angle ∢AOB. By definition, we know that the measure of the central angle is the measure of its intercepted arc. In this case,

$$m\widehat{AB} = m\measuredangle AOB \text{ or } m\measuredangle AOB = 80°$$

Radius $\overline{OA}$ and radius $\overline{OB}$ are congruent and form two sides of △OAB. By a theorem, the angles opposite these two congruent sides must, themselves, be congruent. Therefore, $m\measuredangle A = m\measuredangle B$.

The sum of the measures of the angles of a triangle is 180°. Therefore,

$$m\measuredangle A + m\measuredangle B + m\measuredangle AOB = 180°.$$

Since $m\measuredangle A = m\measuredangle B$, we can write

$$m\measuredangle A + m\measuredangle A + 80° = 180° \text{ or}$$

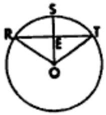
$$2m\measuredangle A = 100° \text{ or}$$

$$m\measuredangle A = 50°.$$

Therefore, the measure of $\measuredangle A$ is 50°

• PROBLEM 7-4

In circle O, in the figure shown, $\overline{OS} \perp \overline{RT}$. Prove $\widehat{RS} \cong \widehat{ST}$.



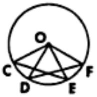
Solution: We can prove the two arcs congruent by proving that their central angles are congruent. Congruent central angles intercept congruent arcs. The central angles of $\widehat{RS}$ and $\widehat{ST}$ are $\measuredangle ROS$ and $\measuredangle SOT$, respectively.

These two angles are contained, respectively, in ΔOER and ΔOET. By proving these two triangles to be congruent to each other, we can conclude that the required central angles are congruent to each other.

Statement	Reason
1. $\overline{OS} \perp \overline{RT}$	1. Given.
2. $\measuredangle OER$ and $\measuredangle OET$ are right angles.	2. Perpendicular lines intersect in right angles.
3. $\measuredangle OER \cong \measuredangle OET$	3. All right angles are congruent.
4. $\overline{OR} \cong \overline{OT}$	4. Radii of the same circle are congruent.
5. $\overline{OE} \cong \overline{OE}$	5. Reflexive property of congruence.
6. $\Delta OER \cong \Delta OET$	6. Hypotenuse-Leg Theorem.
7. $\measuredangle ROS \cong \measuredangle SOT$	7. Corresponding angles of congruent triangles are congruent.
8. $\widehat{RS} \cong \widehat{ST}$	8. Congruent central angles of the same circle intercept congruent arcs.

• PROBLEM 7-5

In the accompanying figure, if $\angle COD \cong \angle FOE$, prove that $\overline{CE} \cong \overline{DF}$.



Solution: In this problem, we will make use of the theorem that congruent chords always intercept congruent arcs in the same circle. The converse of this is also true: congruent arcs have congruent chords.

Statement	Reason
1- $\angle COD \cong \angle FOE$	1- Given.
2- $\overset{\frown}{CD} \cong \overset{\frown}{FE}$	2- Congruent central angles intercept congruent arcs.
3- $\overset{\frown}{DE} \cong \overset{\frown}{DE}$	3- Reflexive property of congruence.
4- $\overset{\frown}{CD} + \overset{\frown}{DE} \cong \overset{\frown}{FE} + \overset{\frown}{DE}$ or $\overset{\frown}{CE} \cong \overset{\frown}{FD}$	4- Congruent quantities added to congruent quantities result in congruent quantities.
5- $\overline{CE} \cong \overline{FD}$	5- Congruent arcs have congruent chords.

INSCRIBED ANGLES AND ARCS

• PROBLEM 7-6

In the circle shown, if $m\overset{\frown}{PR} = 70$ and $m\overset{\frown}{QR} = 80$, find $m\angle P$, $m\angle Q$, and $m\angle R$.

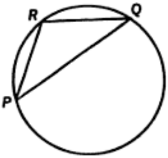


Figure 1

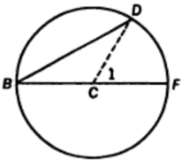


Figure 2

Solution: By definition, the three angles whose measure we are asked to determine are inscribed angles. We will prove the theorem which states that the measure of an inscribed angle is equal to one-half the measure of the intercepted arc. This theorem can be proved in three cases. One where the center of the circle lies on a side of the angle, one where it lies in the interior of the angle, and one where it is assumed to be on the exterior of the angle.

Case 1: We have a circle with C as the center and $\measuredangle DBF$ inscribed so that $\overline{BF}$ is a diameter (figure 2). We draw radius $\overline{CD}$. Since they are radii, $\overline{BC} \cong \overline{CD}$. In ΔBCD, $\measuredangle B \cong \measuredangle D$ because when a triangle has two sides congruent, the angles opposite those sides are congruent. $\measuredangle 1$ and $\measuredangle C$ are, by definition, supplementary. Therefore, $m\measuredangle 1 + m\measuredangle C = 180^\circ$. Since they are the three angles of ΔBCD,

$$m\measuredangle D + m\measuredangle B + m\measuredangle C = 180.$$

Therefore, by transitivity,

$$m\measuredangle D + m\measuredangle B = m\measuredangle 1.$$

Since $\measuredangle 1$ is a central angle, $m\measuredangle 1 = m\overset{\frown}{DF}$ and

$$2(m\measuredangle B) = m\overset{\frown}{DF} \text{ and}$$

$$m\measuredangle B = \frac{1}{2}\, m\overset{\frown}{DF}.$$

Case 1 is now established and will be instrumental in proving Cases 2 and 3.

Case 2: Now C is in the interior of $\measuredangle DBF$ (figure 3). If we draw diameter $\overline{AB}$, as shown, then $m\measuredangle DBF = m\measuredangle 1 + m\measuredangle 2$. Angles 1 and 2 are both inscribed angles with one side of each containing the center of the circle and, as such, their measures can be found by referring to case 1.

$$m\measuredangle 1 = \frac{1}{2} m\overset{\frown}{FA} \quad \text{and } m\measuredangle 2 = \frac{1}{2}\, m\overset{\frown}{AD}$$

$$\begin{aligned} m\measuredangle 1 + m\measuredangle 2 &= \frac{1}{2}\, m\overset{\frown}{FA} + \frac{1}{2}\, m\overset{\frown}{AD} \\ &= \frac{1}{2}\, (m\overset{\frown}{FA} + m\overset{\frown}{AD}). \end{aligned}$$

As the figure shows, $\overset{\frown}{FA}$ and $\overset{\frown}{AD}$ compose $\overset{\frown}{FD}$.

Therefore, $m\overset{\frown}{FA} + m\overset{\frown}{AD} = m\overset{\frown}{FD}$.

$$m\measuredangle 1 + m\measuredangle 2 = \frac{1}{2}\, m\overset{\frown}{FD}.$$

By substitution, then, $m\measuredangle DBF = \frac{1}{2}\, m\overset{\frown}{FD}$.

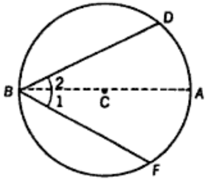


Figure 3

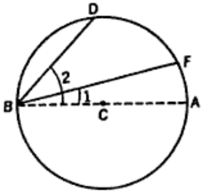


Figure 4

Case 3: Here, C is in the exterior of the inscribed angle (figure 4), but a technique similar to case 2 is used. Introduce the diameter to from an angle like that covered by case 1.

$$m\sphericalangle DBF = m\sphericalangle 2 - m\sphericalangle 1 .$$

But, by the case 1 result, since $\sphericalangle 2$ and $\sphericalangle 1$ both have one side that includes the circle's center,

$$m\sphericalangle 2 = \frac{1}{2} m\overset{\frown}{DA} \quad \text{and} \quad m\sphericalangle 1 = \frac{1}{2} m\overset{\frown}{FA}.$$

By substitution, $m\sphericalangle DBF = \frac{1}{2} m\overset{\frown}{DA} - \frac{1}{2} m\overset{\frown}{FA}$

$$= \frac{1}{2}(m\overset{\frown}{DA} - m\overset{\frown}{FA}).$$

Since $\overset{\frown}{DA}$ is composed of $\overset{\frown}{DF}$ and $\overset{\frown}{FA}$, the $m\overset{\frown}{DA} - m\overset{\frown}{FA} = m\overset{\frown}{DF}$.

By substitution, $m\sphericalangle DBF = \frac{1}{2} m\overset{\frown}{DF}$.

Now we can use the theorem that states if an angle is inscribed, its measure is equal to $\frac{1}{2}$ the measure of the intercepted arc, to answer the original question.

We were told that $m\overset{\frown}{PR} = 70$ and $m\overset{\frown}{QR} = 80$. Therefore, since a circle contains a measure of 360, $m\overset{\frown}{PQ} = 360 - (70+80)$ or $m\overset{\frown}{PQ} = 210$. $\sphericalangle P$, $\sphericalangle Q$, and $\sphericalangle R$ are all inscribed angles. This allows us to find their measure by using the theorem we have just proved. $\sphericalangle P$ intercepts $\overset{\frown}{QR}$, $\sphericalangle Q$ intercepts $\overset{\frown}{PR}$, and $\sphericalangle R$ intercepts $\overset{\frown}{PQ}$.

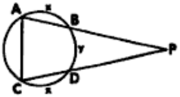
Therefore, $m\sphericalangle P = \frac{1}{2} m\overset{\frown}{QR} = \frac{1}{2}(80) = 40$

$$m\sphericalangle Q = \frac{1}{2} m\overset{\frown}{PR} = \frac{1}{2}(70) = 35$$

$$m\sphericalangle R = \frac{1}{2} m\overset{\frown}{PQ} = \frac{1}{2}(210) = 105.$$

• PROBLEM 7-7

In a circle, as shown in the accompanying figure, congruent chords $\overline{AB}$ and $\overline{CD}$ are extended through B and D, respectively, until they intersect at P. Prove that triangle APC is an isosceles triangle.



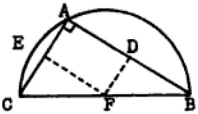
Solution: To show that ΔAPC is isosceles, we must show $\overline{PA} \cong \overline{PC}$. This can be done by showing $\measuredangle A \cong \measuredangle C$.

The proof of this last point will involve using the theorem which states that congruent chords have arcs of equal measure. This, along with the reflexive property of arc measure, will show that $\measuredangle A$ and $\measuredangle C$ are inscribed in arcs of equal measure, which implies the necessary congruence condition.

Statement	Reason
1. Chord $\overline{AB} \cong$ Chord $\overline{CD}$.	1. Given.
2. $m\overset{\frown}{BD} = m\overset{\frown}{BD} = y$	2. Reflexive Property of Equality.
3. $m\overset{\frown}{AB} = m\overset{\frown}{CD} = x$	3. In a circle, congruent chords have equal arcs.
4. $m\overset{\frown}{ABD} = x + y$ and $m\overset{\frown}{BDC} = x + y$	4. The measure of an arc is equal to the sum of the measures of its parts.
5. $m\overset{\frown}{ABD} = m\overset{\frown}{BDC}$	5. Transitive Property of Equality
6. $\measuredangle A \cong \measuredangle C$	6. In a circle, if inscribed angles intercept equal arcs, the angles are congruent.
7. $\overline{PA} \cong \overline{PC}$	7. In a triangle, if two angles are congruent, the sides opposite these angles are congruent.
8. ΔAPC is an isosceles triangle	8. An isosceles triangle is a triangle that has two congruent sides.

• PROBLEM 7-8

Let $m\measuredangle A = 90°$ in ΔABC. Let D, E, and F be the midpoints of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively. Prove that F is the center of a semicircle which contains B, A, and C.

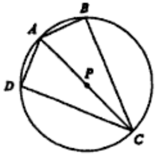


Solution: The figure shows $\triangle ABC$ and part of the circle on which A, B, and C lie. To show that F is the center of the circle, we will prove that $\overline{FE}$ and $\overline{FD}$ are perpendicular bisectors of chords $\overline{CA}$ and $\overline{AB}$. Since the perpendicular bisectors of the chords of a circle meet at the center of the circle, we will be able to show that F is the center of the circle passing through A, B, and C. Since an angle inscribed in a semi-circle is a right angle, and $\angle A$ is a right angle inscribed in a circle, we will be able to conclude that $\widehat{CAB}$ is a semi-circle, and F is its center.

First, note that E and D are the midpoints of $\overline{CA}$ and $\overline{AB}$, as the problem states. Hence, $\overline{EF} \parallel \overline{AB}$ and $\overline{DF} \parallel \overline{AC}$. Since the corresponding angles of 2 parallel lines cut by a transversal are congruent, $\angle CEF \cong \angle CAD$ and $\angle BDF \cong \angle BAC$. Therefore, $\overline{FE} \perp \overline{CA}$ and $\overline{FD} \perp \overline{AB}$. Because E bisects $\overline{CA}$ and D bisects $\overline{AB}$, $\overline{FE}$ and $\overline{FD}$ are the perpendicular bisectors of chords $\overline{CA}$ and $\overline{AB}$. As explained above, the perpendicular bisectors of the chords of a circle intersect at the center of the circle. Hence, F is the center of the circle containing points A, B, and C. (See figure.) Furthermore, $\angle A$ is inscribed in circle F and has measure 90°. Hence, $\angle A$ must intercept a semi-circle i.e. CAB is a semi-circle, with center at F.

• PROBLEM 7-9

Given: Points A, B, C, and D are in $\odot P$; $\widehat{AB} \cong \widehat{AD}$; $\widehat{BC} \cong \widehat{DC}$.
Prove: $\angle B \cong \angle D$.



Solution: There are two ways of solving this problem. The first is to prove $\triangle ADC \cong \triangle ABC$ and then, $\angle B \cong \angle D$ follows by corresponding parts. The second method is to show $\angle B$ and $\angle D$ both intercept equal arcs.

Method 1: In the same circle, or in congruent circles, if two arcs are congruent, then their chords are congruent. Therefore, $\widehat{AD} \cong \widehat{AB}$, which is given, implies $\overline{AD} \cong \overline{AB}$ and $\widehat{BC} \cong \widehat{DC}$, which is given, implies $\overline{BC} \cong \overline{DC}$. $\triangle ADC \cong \triangle ABC$ follows by the SSS Postulate.

Statements	Reasons
1. Points A, B, C, and D are in $\odot P$; $\widehat{AB} \cong \widehat{AD}$; $\widehat{BC} \cong \widehat{DC}$	1. Given.

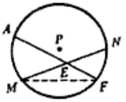
2. $\overline{AB} \cong \overline{AD}$; $\overline{BC} \cong \overline{DC}$	2. In the same circle, congruent arcs intercept congruent chords.
3. $\overline{AC} \cong \overline{AC}$	3. Reflexivity Property.
4. $\triangle ABC \cong \triangle ADC$	4. The SSS Postulate.
5. $\angle B \cong \angle D$	5. Definition of congruenct triangles.

Method 2:

Statements	Reasons
1. Points A, B, C, and D in $\odot P$; $\overset{\frown}{AB} \cong \overset{\frown}{AD}$; $\overset{\frown}{BC} \cong \overset{\frown}{DC}$	1. Given
2. $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{AD} + m\overset{\frown}{DC}$	2. Addition Postulate.
3. $m\overset{\frown}{ABC} = m\overset{\frown}{ADC}$	3. Substitution.
4. $\overset{\frown}{ABC} \cong \overset{\frown}{ADC}$	4. Definition of Congruent Arcs.
5. $\angle B \cong \angle D$	5. Inscribed angles that intercept congruent arcs are congruent.

• PROBLEM 7-10

Given two intersecting chords of a circle, show that the measure of the angle formed by the intersection is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.



Solution: We are required to relate $\angle AEM$ to $\overset{\frown}{AM}$ and $\overset{\frown}{NF}$. Here, by drawing $\overline{MF}$, we can express $\overset{\frown}{AM}$ and $\overset{\frown}{NF}$ in terms of inscribed angles $\angle NMF$ and $\angle AFM$. $\angle NEF$ can also be expressed in terms of these two angles, since $\angle NEF$ is an exterior angle of $\triangle MEF$. Thus, we solve for $\angle NEF$ in terms of $\angle NMF$ and $\angle AFM$. Then we solve for $\angle NMF$ in terms of $\overset{\frown}{NF}$ and for $\angle AFM$ in terms of $\overset{\frown}{AM}$. Substituting into the expression for $\angle NEF$, we will then have an equation for $\angle NEF$ in terms of the intercepted arcs.

Given: $\overline{AF}$ and $\overline{MN}$ are chords of $\odot P$ that intersect at E.

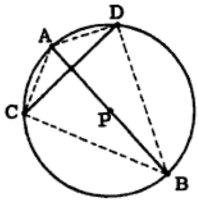
Prove: $m\angle NEF = \frac{1}{2}(m\overset{\frown}{AM} + m\overset{\frown}{NF})$

Statements	Reasons
1. $\overline{AF}$ and $\overline{MN}$ are chords of ⊙P that intersect at E.	1. Given.
2. $m\sphericalangle AFM = \frac{1}{2}m\widehat{AM}$ $m\sphericalangle NMF = \frac{1}{2}m\widehat{NF}$	2. The measure of the inscribed angle is equal to one-half the measure of the intercepted arc.
3. $\sphericalangle NEF$ is an exterior angle of ΔMEF	3. An exterior angle of a triangle is an angle that forms a linear pair with one of the interior angles of the angle.
4. $m\sphericalangle NEF = m\sphericalangle AFM + m\sphericalangle NMF$	4. The measure of the exterior angle of a triangle equals the sum of the measure of the two remote interior angles.
5. $m\sphericalangle NEF = \frac{1}{2}m\widehat{AM} + \frac{1}{2}m\widehat{NF}$ or $m\sphericalangle NEF = \frac{1}{2}(m\widehat{AM} + m\widehat{NF})$	5. Substitution Postulate (Steps 2 and 4)

CHORDS

• PROBLEM 7-11

Prove that the line containing the midpoints of the major and minor arcs of a chord of a circle is the perpendicular bisector of the chord.



Solution: A perpendicular bisector is a line, all of whose points are equidistant from the endpoints of the given segment. We show that two points, the midpoints of the major and minor arcs, are equidistant from the endpoints of the chord and thus that the line determined by these points must be the perpendicular bisector. To show A and B, the midpoints of the arcs, are equidistant from chord endpoints C and D, we show $\overline{AC} \cong \overline{AD}$ and $\overline{CB} \cong \overline{DB}$. However, from the given $\widehat{AC} \cong \widehat{AD}$ and $\widehat{CB} \cong \widehat{BD}$, and the congruence of the segments follows.

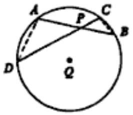
Given: $\overline{CD}$ is a chord of ⊙P. A is the midpoint of the minor arc $\widehat{CD}$. B is the midpoint of major arc $\widehat{CD}$.

Prove: $\overleftrightarrow{AB}$ is the perpendicular bisector of $\widehat{CD}$.

Statements	Reasons
1. A is the midpoint of minor arc $\widehat{CD}$. B is the midpoint of major arc $\widehat{CD}$.	1. Given
2. $m\widehat{AC} = m\widehat{AD} = 1/2\ m\widehat{CD}$ $m\widehat{BC} = m\widehat{BD} = \frac{1}{2}m\widehat{CD}$	2. The midpoint of an arc divides the arc into two congruent arcs.
3. $\overline{AC} \cong \overline{AD}$; $\overline{CB} \cong \overline{BD}$	3. If two arcs are congruent, their chords are also congruent.
4. A is equidistant from C and D. B is equidistant from C and D.	4. Definition of equidistance.
5. $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{BC}$.	5. If a line contains two points equidistant from the endpoints of the segment, then the line is the perpendicular bisector of the segment.

• PROBLEM 7-12

Two chords intersect in the interior of a circle, thus determining two segments in each chord. Show that the product of the length of the segments of one chord equals the product of the lengths of the segments of the other chord.



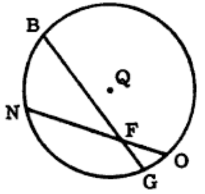
Solution: Restating the above problem, in terms of the figure, we obtain $AP \cdot PB = DP \cdot PC$. We can rewrite this (divide both sides by PB · DP) as $\frac{AP}{DP} = \frac{PC}{PB}$. To prove the proportionality, we show that two triangles are similar. In this case, $\triangle APD \sim \triangle CPB$ by the A-A Similarity Theorem.

Given: Chords $\overline{AB}$ and $\overline{CD}$ of $\odot Q$ intersect at point P.
Prove: AP · PB = DP · PC.

Statements	Reasons
1. Chords $\overline{AB}$ and $\overline{CD}$ of $\odot Q$ intersect at point P.	1. Given.
2. $\sphericalangle ADC \cong \sphericalangle ABC$.	2. Inscribed angles that cut the same arc are congruent.
3. $\sphericalangle APD \cong \sphericalangle CPB$.	3. Vertical angles are congruent.
4. $\triangle APD \sim \triangle CPB$.	4. The A-A Similarity Theorem.
5. $\frac{AP}{PD} = \frac{PC}{PB}$	5. The sides of similar triangles are proportional.
6. AP · PB = PC · PD.	6. In a proportion, the product of the means must equal the product of the extremes.

• PROBLEM 7-13

In the figure below, if $m\widehat{BN} = 50$ and $m\sphericalangle OFG = 35$, find $m\widehat{OG}$.



Solution: We know that given two intersecting chords in a circle, the measure of the angle formed by the intersection is equal to one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle. Here, $\widehat{BN}$ and $\widehat{OG}$ are the intercepted arcs of $\sphericalangle OFG$ and its vertical angle. Therefore,

$$\text{(i)}\quad m\sphericalangle OFG = \tfrac{1}{2}(m\widehat{BN} + m\widehat{OG}).$$

Substituting in the values $m\widehat{BN} = 50$ and $m\angle OFG = 35$, we obtain:

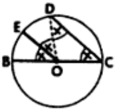
$$\text{(ii)}\quad 35 = \frac{1}{2}(50 + m\widehat{OG})$$

$$\text{(iii)}\quad (2)35 = 50 + m\widehat{OG}$$

$$\text{(iv)}\quad m\widehat{OG} = (2)35 - 50 = 70 - 50 = 20$$

• PROBLEM 7-14

In the accompanying figure, diameter $\overline{BC}$ is drawn in circle O. Radius $\overline{OE}$ is drawn parallel to chord $\overline{CD}$. Prove that the measure of arc BE is equal to the measure of arc ED, i.e. $\widehat{BE} \cong \widehat{ED}$.



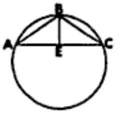
Solution: We know that in any given circle, arcs of equal measure have central angles of equal measure. Radius $\overline{OD}$, when drawn, becomes the transversal of parallel segments $\overline{OE}$ and $\overline{CD}$. Diameter $\overline{CB}$ is also a transversal of these lines. If we prove $m\angle EOD \cong m\angle BOE$, then we have shown that the arcs which these two central angles intercept are also equal.

Statement	Reason
1. Draw radius $\overline{OD}$	1. One and only one straight line may be drawn through two points.
2. $\overline{OE} \parallel \overline{CD}$.	2. Given.
3. $m\angle EOD = m\angle CDO$.	3. If two parallel lines are cut by a transversal, the alternate interior angles are congruent and of equal measure.
4. $\overline{OD} \cong \overline{OC}$	4. Radii of a circle are congruent.
5. $m\angle BOE = m\angle OCD$.	5. If two parallel lines are cut by a transversal, the corresponding angles are congruent and of equal measure.
6. $m\angle OCD = m\angle CDO$.	6. If two sides of a triangle are congruent, the angles

	opposite these sides are congruent and of equal measure.
7. m$\angle$BOE = m$\angle$CDO.	7. Transitive property of equality.
8. m$\angle$BOE ≅ m$\angle$EOD.	8. Same as reason 7.
9. $\overset{\frown}{BE} \cong \overset{\frown}{ED}$.	9. In a circle, central angles whose measures are equal intercept congruent arcs.

• PROBLEM 7-15

In the circle shown, $\overline{BE} \perp \overline{AC}$ and B is the midpoint of arc $\overset{\frown}{AC}$. Prove $\overline{AE} \cong \overline{CE}$.



Solution: In this proof, we must use the theorem that congruent arcs have congruent chords. To show $\overline{AE} \cong \overline{CE}$, we first prove $\triangle ABE \cong \triangle CBE$. To show $\triangle ABE \cong \triangle CBE$, we use the Hypotenuse Leg Theorem. Right angles BEA and BEC are congruent. $\overline{BE} \cong \overline{BE}$. For the third congruence, we note that $\overline{AB}$ and $\overline{BC}$ are chords that intersect congruent arcs. From the above theorem, it follows that $\overline{AB} \cong \overline{BC}$.

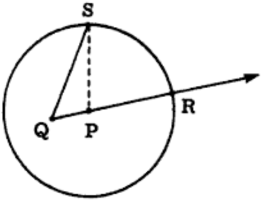
Statement	Reason
1. $\overline{BE} \perp \overline{AC}$	1. Given.
2. $\angle$BEA and $\angle$BEC are right angles.	2. Perpendicular lines intersect to form right angles.
3. $\triangle$ABE and $\triangle$BCE are right triangles.	3. A triangle containing a right angle is a right triangle.
4. B is the midpoint of $\overset{\frown}{AC}$.	4. Given.
5. $\overset{\frown}{AB} \cong \overset{\frown}{BC}$.	5. A midpoint of an arc divides the arc into two congruent sections.
6. $\overline{AB} \cong \overline{BC}$	6. Congruent arcs have congruent chords.
7. $\overline{BE} \cong \overline{BE}$	7. Reflexive property of congruence.

8. $\Delta ABE \cong \Delta CBE$.	8. hy. leg $\cong$ hy. leg.
9. $\overline{AE} \cong \overline{CE}$.	9. Corresponding sides of congruent triangles are congruent.

• PROBLEM 7-16

Given: Point Q is in the interior of $\odot P$; $\overrightarrow{QP}$ intersects $\odot P$ in R; S is any point on $\odot P$.

Prove: $RQ > SQ$ (Hint: Use $\overline{SP}$).



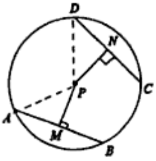
Solution: We must show $RQ > SQ$. Since $RQ = RP + PQ$, this is the same as proving $RP + PQ > SQ$.

To relate SQ and PQ, we use the Triangle Inequality Theorem - that is, in Δ SPQ, $SQ < QP + SP$. Since $\overline{SP}$ and $\overline{PR}$ are radii, $\overline{SP} \cong \overline{PR}$ and $QP + SP = QP + PR$. The inequality then becomes $SQ < QP + PR$ or $SQ < RQ$.

Statements	Reasons
1. (see problem statement)	1. Given
2. $QR = QP + PR$	2. Definition of betweenness.
3. $SQ < QP + PS$	3. Triangle Inequality Theorem.
4. $SP = PR$	4. All radii of a circle are congruent.
5. $SQ < QP + PR$	5. Substitution Postulate (Step 3,4)
6. $SQ < QR$	6. Substitution Postulate (Step 5,2)

• PROBLEM 7-17

Given: $\overline{AB}$ and $\overline{CD}$ are chords of circle P; $\overline{PM} \perp \overline{AB}$ at M; $\overline{PN} \perp \overline{CD}$ at N; $\overline{PM} \cong \overline{PN}$. Prove: $\overline{AB} \cong \overline{CD}$.



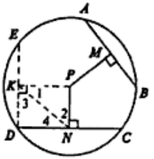
Solution: To prove congruence of lines, we must look for congruent triangles. Here, in drawing the radii $\overline{AP}$ and $\overline{DP}$, we make use of the fact that all radii of a given circle are congruent. After proving $\Delta AMP \cong \Delta DNP$, we have $\overline{AM} \cong \overline{DN}$. We could then repeat the procedure, drawing radii $\overline{PC}$ and $\overline{PB}$, and prove $\overline{NC} \cong \overline{MB}$, but that is unnecessary. The line containing the center of the circle and perpendicular to any chord bisects that chord. Therefore, if $\overline{AM} \cong \overline{DN}$; $\overline{AB}$ (which equals $2 \cdot AM$) must be congruent to $\overline{CD}$ (which equals $2 \cdot DN$).

Statements	Reason
1. $\overline{AB}$ and $\overline{CD}$ are chords of circle P; $\overline{PM} \perp \overline{AB}$ at M; $\overline{PN} \perp \overline{CD}$ at N; $\overline{PM} \cong \overline{PN}$.	1. Given
2. $AP = DP$.	2. All radii of a circle are equal in measure.
3. $\Delta AMP \cong \Delta DNP$.	3. If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and leg of another right triangle, then the two triangles are congruent.
4. $AM = DN$.	4. Corresponding parts of congruent triangles are congruent.
5. M is the midpoint of $\overline{AB}$; N is the midpoint of $\overline{CD}$.	5. The line containing the center of the circle and perpendicular to the chord intersects the chord at its midpoint.
6. $AB = 2AM$ $CD = 2DN$	6. Property of the midpoint.
7. $AB = CD$.	7. Multiplication Postulate and Substitution Postulate.
8. $\overline{AB} \cong \overline{CD}$	8. Two line segments are congruent if they are of the same length.

• PROBLEM 7-18

Show that in the same circle (or in congruent circles) if two chords are not congruent, then the longer chord is nearer the center of the circle than the shorter chord.

Solution: Consider chords $\overline{AB}$ and $\overline{CD}$ in the accompanying figure. $AB < CD$. We want to show $PM > PN$.



The inequality $AB < CD$ could lead us to an inequality of angles, if $\overline{AB}$ and $\overline{CD}$ were sides of a triangle. Similarly, if $\overline{PM}$ and $\overline{PN}$ were sides of a triangle, then by showing the angle opposite $\overline{PM}$ is greater than the angle opposite $\overline{PN}$, we could reach the desired conclusion.

We can create the necessary triangles by constructing chord $\overline{ED} \cong \overline{AB}$. Consider point K, the midpoint of $\overline{ED}$. Then, $KD = \frac{1}{2}ED$. Also, $DN = \frac{1}{2}DC$.

Since $DC > ED$, then $DN > KD$. In triangle ΔDKN, $m\measuredangle 3 > m\measuredangle 4$. Because $\measuredangle DKP$ and $\measuredangle DNP$ are right angles, $m\measuredangle 3 + m\measuredangle 1 = m\measuredangle 4 + m\measuredangle 2 = 90^\circ$. Since $m\measuredangle 3 > m\measuredangle 4$, $m\measuredangle 1 < m\measuredangle 2$. $PK > PN$ and the distance of chord $\overline{AB}$ from the center, $PM = PK$, is greater than the distance of the longer chord $\overline{DC}$ from the center, $\overline{PN}$.

Given: $\overline{AB}$ and $\overline{CD}$ are chords of $\odot P$; $\overline{PM} \perp \overline{AB}$ at M;

$\overline{PN} \perp \overline{CD}$ at N; $CD > AB$.

Prove: $PN < PM$.

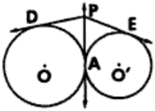
Statements	Reasons
1. $\overline{AB}$ and $\overline{CD}$ are chords of $\odot P$; $\overline{PM} \perp \overline{AB}$ at M; $\overline{PN} \perp \overline{CD}$ at N; $CD > AB$	1. Given.
2. Construct chord $\overline{ED}$ such that $ED = AB$ and E and P are on the same side of DC.	2. From any point on the circle, a chord can be constructed of a given length, provided the length is less than the diameter.
3. K is the midpoint of $\overline{ED}$	3. Every line segment $\overline{AB}$ has one and only one point C such that $AC = CB$.
4. $KD = \frac{1}{2}ED$	4. Definition of the midpoint of a segment.
5. $\overline{PK} \perp \overline{ED}$	5. The line joining the center of the circle and the midpoint of a chord is the perpendicular bisector of the chord.
6. $PK = PM$	6. In the same circle (or congruent circles), chords are congruent if and only if they are equidistant from the center of the circle.
7. $DN = \frac{1}{2}DC$	7. The perpendicular from the center of the circle to a chord bisects the chord.
8. $DC > ED$	8. Substitution Postulate
9. $DN > KD$	9. Halves of unequal quantities are unequal in the same sense,
10. $m\measuredangle 3 > m\measuredangle 4$	10. In a triangle, if the length of the side opposite the first angle is greater than the length of the side opposite the second angle, the first

	angle is greater in measure than the second angle.
11. $m\angle 3 + m\angle 1 = 90^\circ$ $m\angle 4 + m\angle 2 = 90^\circ$	11. If two angles form a right angle, thei measures sum to 90°.
12. $m\angle 1 = 90^\circ - m\angle 3$ $m\angle 2 = 90^\circ - m\angle 4$	12. Equals subtracted from equals leave equals.
13. $90^\circ - m\angle 4 > 90^\circ - m\angle 3$	13. If $a = b$ and $c < d$, then $a - c > b - d$. (The equality is $90^\circ = 90^\circ$; the inequality comes from Step 10.)
14. $m\angle 2 > m\angle 3$	14. Substitution Postulate.
15. $PK > PN$	15. In a triangle, if the measure of the angle opposite the first side is greater than the measure of the angle opposite the second side, then the length of the first side is greater than the length of the second.
16. $PM > PN$	16. Substitution Postulate (Steps 15 and 6)

TANGENTS, LINES AND CIRCLES

• PROBLEM 7-19

Prove that if two circles are tangent externally, tangents drawn to the circles from any point on their common internal tangent are equal in length. (See figure)



Solution: A common internal tangent is a line which is tangent to both circles and intersects their line of center. The line of center has endpoints O and O'. $\overleftrightarrow{PA}$ is the common internal tangent of circles O and O', and $\overleftrightarrow{PD}$ and $\overleftrightarrow{PE}$ are external tangents drawn from a point P on the common internal tangent. If both $\overleftrightarrow{PE}$ and $\overleftrightarrow{PD}$ can be shown to be equal in length to the same line segment, then we can conclude that they are equal to each other.

Statement	Reason
1. $\overleftrightarrow{PD}$ and $\overleftrightarrow{PA}$ are tangents to circle O.	1. Given
2. $PD = PA$.	2. The lengths of tangents drawn to a circle from an external point are equal.
3. $\overleftrightarrow{PE}$ and $\overleftrightarrow{PA}$ are tangents to circle O'.	3. Given.

4. PA = PE.	4. Same as Reason 2.
5. If PD = PA and PE = PA, then PD = PE.	5. Transitive Property of Equality.

• PROBLEM 7-20

A belt moves over two wheels which are the same size and crosses itself at right angles as shown in the figure. A,B,C, and D are points of tangency; E is the intersection of the tangents; and O is the center of one of the circles. The radius of each wheel is 7 inches.

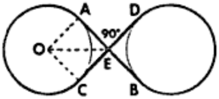
(a) Show that m∢AOE = 45°.

(b) Find

(1) the length of $\overline{AE}$

(2) the length of major arc $\widehat{AC}$. [Use $\pi = \frac{22}{7}$]

(3) the length of the entire belt.



Solution: (a) We will first show that AOCE is a square in order to prove that ∢AOC is a right angle. Then we will show that diagonal $\overline{OE}$ bisects ∢AOC. If m∢AOC = 90°, then ∢AOE must equal 1/2 of 90° or 45°.

Statement	Reason
1. $\overline{OA} \cong \overline{OC}$	1. Radii of a circle are congruent.
2. $\overline{AE}$ and $\overline{CE}$ are tangents at points A and C respectively.	2. Given.
3. $\overline{OA} \perp \overline{AE}$ and $\overline{OC} \perp \overline{CE}$	3. If a line is tangent to a circle, the line is perpendicular to the radius drawn to the point of contact.
4. ∢OAE and ∢OCE are right angles.	4. Perpendicular lines intersect to form right angles.
5. ∢AEC is a right angle.	5. Given.
6. ∢OAE is supplementary to ∢AEC.	6. Any two right angles are supplementary.

7. $\overline{OA} \parallel \overline{EC}$	7. If interior angles on the same side of a transversal are supplementary, then the lines are parallel.
8. ∢OCE is supplementary to ∢AEC.	8. Any two right angles are supplementary.
9. $\overline{OC} \parallel \overline{AE}$	9. Same as reason 7.
10. AOCE is a square.	10. A quadrilateral in which (1) all opposite sides are parallel; (2) there exists two adjacent sides which are congruent; and (3) at least one vertex angle is a right angle is a square.
11. ∢AOC is a right angle.	11. All vertex angles of a square are right angles.
12. m∢AOC = 90°	12. Right angles measure 90°.
13. $\overline{OE}$ bisects ∢AOC	13. The diagonal of a square bisects the vertex angles through which it is drawn.
14. m∢AOC = 2m∢AOE	14. An angle bisector divides the angle into two angles of equal measure.

Statement	Reason
15. 2(m∢AOE) = 90°.	15. Substitution Postulate.
16. m∢AOE = 45°.	16. Division Property of Equality.

(b) 1. Since all four sides of a square are of equal measure, AE = OC. $\overline{OC}$ is a radius (r) given to be 7 in. in length.

Therefore, AE = 7 in.

2. Since central angle AOC measures 90°, minor arc $\widehat{AC}$ = 90°. Therefore, major arc $\widehat{AC}$ = 360 - 90 or 270.
arc length = $\dfrac{\text{degree measure of the arc}}{360}$ x Circumference.
Circumference = C = $2\pi r$, where r is the radius. By substitution, $C = 2\pi(7) = 14\pi$. Using $\pi = \frac{22}{7}$, we obtain $C = \frac{14 \times 22}{7}$ = 44 in. Therefore, length of major arc $\widehat{AC} = \frac{270}{360}(44) = \frac{3}{4}(44)$ = 33. Therefore, length of major arc $\widehat{AC}$ = 33 in.

3. From the diagram, we find that length of entire belt = length of major arc $\widehat{AC}$ + $\overline{AE}$ + $\overline{EC}$ + $\overline{ED}$ + $\overline{EB}$ + length of major arc $\widehat{BD}$. We know that $\overline{AE} = \overline{EC}$ = 7 inches , and that the length of major arc $\widehat{AC}$ = 33 inches.

Using the same manner we employed to find the above measurements, we can derive $\overline{ED}$, $\overline{EB}$ and length of major arc $\widehat{DB}$.

If we drew the radii from D and B to the center of the circle on the right, they would form a square symmetric to AOCE and, as such, $\overline{ED} = \overline{EB}$ = radius of that circle. We are given the radius = 7 in. for both circles. Therefore, $\overline{ED} = \overline{EB} = 7$ in.

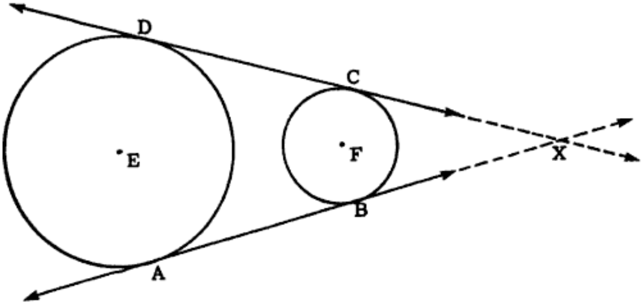
By a similar symmetry argument, we prove major arc $\overset{\frown}{BD}$ has the same length of major arc $\overset{\frown}{AC}$. Therefore, major arc $\overset{\frown}{BD} = 33$ in.

By substitution, length of entire belt = $33 + 7 + 7 + 7 + 7 + 33 = 94$

Therefore, the length of the entire belt = 94 in.

• PROBLEM 7-21

Given: Circles E and F; $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ tangent to both circles at A, B, C, and D. Prove: $\overline{AB} \cong \overline{CD}$.



Solution: Extend $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ to the right until they intersect at a point X.

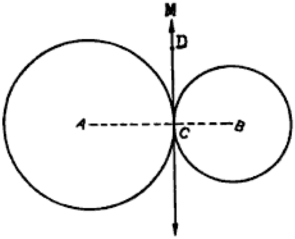
Since A, B, C, and D are points of tangency, $\overleftrightarrow{AX}$ and $\overleftrightarrow{DX}$ will be tangent to ⊙E and $\overleftrightarrow{BX}$ and $\overleftrightarrow{CX}$ will be tangents to ⊙F. Two tangents drawn to a circle from an external point are of equal length. Therefore, $\overline{AX} \cong \overline{DX}$ and $\overline{BX} \cong \overline{CX}$.

We know that the length of the whole segment is equal to the sum of the lengths of its parts. Hence, $\overline{AX} = \overline{AB} + \overline{BX}$ and $\overline{DX} = \overline{DC} + \overline{CX}$. Accordingly, $\overline{AB} = \overline{AX} - \overline{BX}$ and $\overline{DC} = \overline{DX} - \overline{CX}$. Since $\overline{AX} \cong \overline{DX}$ and $\overline{BX} \cong \overline{CX}$, we can conclude that $\overline{AB} \cong \overline{DC}$.

• PROBLEM 7-22

Circles A and B are externally tangent at C. Prove that A, C, and B are collinear.

Solution: If two adjacent angles are right angles, then their non-common sides form a straight angle. Since the rays of a straight angle form a straight line, any points on these rays are collinear.



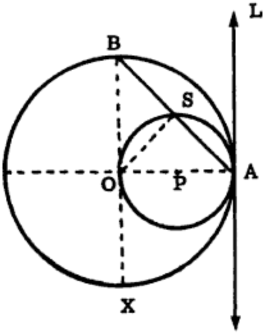
C is the point of tangency of line m and circle B. Since radii of a circle are perpendicular to tangent lines at the points of tangency, therefore, radius $\overline{CB}$ is perpendicular to line m. Accordingly, radius $\overline{AC}$ is also perpendicular to line m.

Since perpendicular lines intersect to form right angles, adjacent angles DCB and DCA are right angles.

Hence, the non-common sides of ∢DCB and ∢DCA form a straight angle, and the points A, C, and B are collinear.

• PROBLEM 7-23

Given two circles with a common tangent at a point A such that the second circle passes through the center of the first. Show that every chord of the first circle that passes through A is bisected by the second circle. (See figure.)

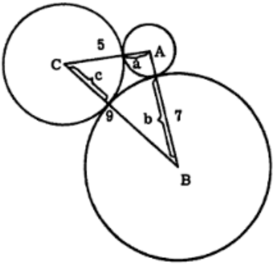


Solution: We are asked to show that if circles P and O are tangent at A, and O lies on circle P, then $\overline{AB}$ is bisected by circle P at point S. We will do this by showing that $\overline{OS}$ is the perpendicular bisector of $\overline{BA}$.

First, note that $\overline{OPA}$ connects the centers of 2 circles tangent at the same point (See figure.) Hence, $\overline{OPA}$ is a straight line passing through the center of circle P. $\overline{OPA}$ is, therefore, a diameter of P, which implies that ∢OSA is inscribed in a semicircle ($\overset{\frown}{OSA}$). This means that ∢OSA is a right angle, and that $\overline{OS} \perp \overline{BA}$. However, any line segment drawn from the center of a circle perpendicular to a chord of the same circle bisects the chord. Since $\overline{OS} \perp \overline{BA}$ at S, $\overline{BS} \cong \overline{SA}$.

• PROBLEM 7-24

Circles A, B, and C are tangent to one another. Find the radii of the three circles if AB = 7, AC = 5, and BC = 9.



Solution: The segment connecting the centers of tangent circles passes through the point of tangency on the circumference of the circles. Thus, the distance to the point of tangency from the center of ⊙C is the radius, c, from the center of ⊙A is the radius, a, and from the center of ⊙B is the radius, b. Therefore, the length of the segments between the centers is the sum of the radii. Hence,

$$AC = c + a$$
$$AB = a + b$$
$$BC = b + c$$

Since AC = 5, AB = 7, and BC = 9, we have

(i) $5 = c + a$
(ii) $7 = a + b$
(iii) $9 = b + c$

Solving for a in terms of c $5 = c + a$, $a = 5 - c$. Substituting this result in the other equations, we obtain:

$$\text{(ii)}\quad 7 = (5-c) + b = 5 + b - c \text{ or } 2 = b - c$$
$$\text{(iii)}\quad 9 = b + c$$

Adding both equations, we eliminate c.

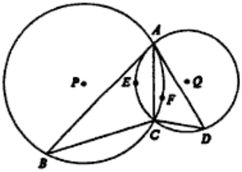
$$11 = 2b \quad\text{or}\quad b = 5.5$$

Since $9 = b + c$, $c = 9 - 5.5 = 3.5$ Hence $a = 5 - c = 5 - 3.5 = 1.5$. Thus, $a = 1.5$, $b = 5.5$, and $c = 3.5$.

INTERSECTING CIRCLES

• PROBLEM 7-25

Given: Circles P and Q intersect at points A and C; chord $\overline{AB}$ of $\odot P$ is tangent to $\odot Q$; chord $\overline{AD}$ of $\odot Q$ is tangent to $\odot P$; E is a point on $\odot Q$; F is a point on $\odot P$. Prove: $\Delta ABC \sim \Delta DAC$ and AC is the mean proportional between BC and DC.



Solution: To show similar triangles, we use either (1) A-A Similarity Theorem, (2) SAS Similarity Theorem, or (3) SSS Similarity Theorem. There are no obvious proportions between the sides. Therefore, we try to find an A-A Similarity. We are to prove $\Delta ABC \sim \Delta DAC$, therefore, $\angle ABC$ corresponds to $\angle CAD$. $\angle B$ is an inscribed angle that intercepts arc AFC and, thus, $m\angle B = \frac{1}{2}m\text{AFC}$. $\angle CAD$ is an angle formed by a chord and a tangent of $\odot P$ and has a measure equal to $\frac{1}{2}$ the intercepted arc or $\frac{1}{2}m\widehat{AFC}$. Therefore, both $m\angle B$ and $m\angle CAD$ equal $\frac{1}{2}m\widehat{AFC}$ and $\angle B \cong \angle CAD$.

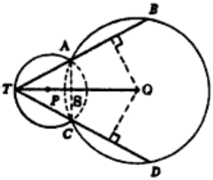
In a similar manner, we show $\angle BAC \sim \angle CDA$. Thus, $\Delta ABC \sim \Delta DAC$. By definition of similar triangles, we show AC is the mean proportional between BC and DC.

Statements	Reasons
1. (See problem statement)	1. Given.

2.	$m\angle B = \frac{1}{2}m\widehat{AFC}$ in ⊙P. $m\angle D = \frac{1}{2}m\widehat{AEC}$ in ⊙Q.	2.	The measure of the inscribed angle equals one-half the measure of the intercepted arc.
3.	$m\angle CAD = \frac{1}{2}m\widehat{AFC}$ in ⊙P. $m\angle BAC = \frac{1}{2}m\widehat{AEC}$ in ⊙Q.	3.	The measure of an angle formed by a chord and a tangent of a circle equals one-half the intercepted arc.
4.	$m\angle B = m\angle CAD$, or, $\angle B \cong \angle CAD$ $m\angle D = m\angle BAC$, or, $\angle D \cong \angle BAC$.	4.	Transitivity Postulate and Definition of Congruence of Angles.
5.	$\Delta ABC \sim \Delta DAC$.	5.	The A-A Similarity Theorem.
6.	$\frac{BC}{AC} = \frac{AC}{DC}$	6.	The sides of similar triangles are proportional.
7.	$AC^2 = BC \cdot DC$.	7.	For a proportion to hold, the product of the means equal the product of the extremes.
8.	AC is the mean proportional between BC and DC.	8.	Defintion of the mean proportional.

• PROBLEM 7-26

Given: Circles P and Q intersect at points A and C; $\overline{PQ}$ meets $\overrightarrow{BA}$ and $\overrightarrow{DC}$ at T, a point of circle P. Prove that $\overline{AB} \cong \overline{CD}$.



Solution: Construct segment $\overline{AC}$ so that it intersects $\overline{PQ}$ at point S. Draw perpendicular line segments from Q to $\overline{AB}$ and $\overline{CD}$. The perpendicular to $\overline{AB}$ intersects $\overline{AB}$ at E, and the perependicular to $\overline{CD}$ intersects $\overline{CD}$ at F. (See figure.)

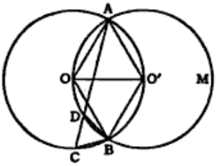
We first prove that $\overline{TPSQ}$ is the perpendicular bisector of $\overline{AC}$. This will enable us to show that $\Delta ATS \cong \Delta CTS$, which implies that $\angle ATS \cong \angle CTS$. We will then have enough data to prove that $\Delta QET \cong \Delta QFT$ or that $\overline{QE} \cong \overline{QF}$. Since chords which are equidistant from the center of a circle are congruent, we will have shown that $\overline{AB} \cong \overline{CD}$.

Circles P and Q intersect at points A and C. Hence, $\overline{QA}$ and $\overline{QC}$ are radii of circle Q, and points A and C are equidistant from point Q. Similarly, A and C are equidistant from point P. Since the locus of points equidistant from two points is the perpendicular bisector of the line segment joining the two points, P and Q must lie on the perpendicular bisector of $\overline{AC}$. Furthermore, since two points determine a line, $\overleftrightarrow{PQ}$ is the perpendicular bisector of $\overline{AC}$. By construction, S and T lie on $\overleftrightarrow{PQ}$. $\overleftrightarrow{TPSQ}$ is the perpendicular bisector of $\overline{AC}$. We use this fact to show that $\triangle ATS \cong \triangle CTS$.

$\overline{AS} \cong \overline{CS}$ and $\angle AST \cong \angle CST$ because $\overleftrightarrow{TPSQ}$ is the perpendicular bisector of $\overline{AC}$. $\overline{TS} \cong \overline{TS}$. By the S.A.S. Postulate, $\triangle ATS \cong \triangle CTS$, and $\angle ATS \cong \angle CTS$. A lies on $\overrightarrow{TE}$, C lies on TF, and $\overleftrightarrow{TPSQ}$ is a straight line. Hence, the last statement can be rewritten as $\angle ETQ \cong \angle FTQ$. By construction, $\angle TEQ \cong \angle TFQ$. $\overline{TQ} \cong \overline{TQ}$ and, by the A.A.S. Postulate, $\triangle QET \cong \triangle QFT$. Hence, $\overline{QE} \cong \overline{QF}$ or $QE = QF$. This means that $\overline{AB}$ and $\overline{CD}$ are equally distant from Q (because $\overline{QE} \perp \overline{AB}$ and $\overline{QF} \perp \overline{CD}$). Therefore, since chords drawn equidistantly from the center of a circle are congruent, $\overline{AB} \cong \overline{CD}$.

• PROBLEM 7-27

Two equal circles are drawn so that the center of each is on the circumference of the other. Their intersection points are A and B. Prove that if, from A, any line is drawn cutting the circles at D and C, then $\triangle BCD$ is equilateral.



Solution: In the accompanying figure, ⊙O and ⊙O' intersect at A and B. $\overleftrightarrow{AD}$ is an arbitrary line that intersects ⊙O' at D and ⊙O at C. To show $\triangle BCD$ is an equilateral triangle, we must either show (1) sides $DC = CB = BD$ or (2) $\angle DCB \cong \angle CBD \cong \angle BDC$. We will show that $m\angle DCB = 60°$ and $m\angle\ BDC = 60°$. Therefore, $m\angle CBD = 180 - m\angle DCB - m\angle BDC = 60$. Thus, we will be able to conclude that $\triangle BCD$ is an equilateral triangle.

To show $m\angle DCB = 60°$, note that $\angle DCB$ is the same angle as $\angle ACB$ which is an inscribed angle of ⊙O. Thus, $m\angle ACB = m\angle DCB = \frac{1}{2} m\ \widehat{AO'B}$.

To find $m\ \widehat{AO'B}$, we consider $\triangle OAO'$ and $\triangle OBO'$. Because equal circles have equal radii, $OA = OO' = O'A$ and $OO' = OB = BO'$. Thus $\triangle OAO'$ and $\triangle OBO'$ are equilateral triangles and $m\angle AOO' = m\angle BOO' = 60°$. $m\angle AOB = m\angle AOO' + m\angle BOO' = 60 + 60 = 120°$. Thus, $m\ \widehat{AO'B} = m\angle AOB = 120°$. Since $m\angle DCB = \frac{1}{2}m\ \widehat{AO'B}$, $m\angle DCB = \frac{1}{2}(120°) = 60°$.

To show $m\angle CDB = 60°$, we show that its supplement, exterior angle $\angle ADB$, has a measure of $120°$. Note that $\angle ADB$ is an inscribed angle of $\odot O'$. Thus, $m\angle ADB = \frac{1}{2}\ \widehat{AMB}$. Since $m\widehat{AMB} + m\widehat{AOB} = 360$, then $m\widehat{AMB} = 360 - m\widehat{AOB} = 360 - 120 = 240$. Substituting, we have $m\angle ADB = \frac{1}{2}(240) = 120$.

Because exterior angle $\angle ADB$ forms a linear pair with $\angle CDB$, $m\angle ADB + m\angle CDB = 180$. Thus, $m\angle CDB = 180 - m\angle ADB = 180 - 120 = 60$.

Finally, in ΔBCD, if $m\angle DCB = 60°$ and $m\angle CDB = 60°$, then $m\angle DBC = 180 - m\angle DCB - m\angle CDB = 180 - 60 - 60 = 60°$.

Thus, all angles of ΔBCD are equal and ΔBCD is an equilateral triangle.

TANGENTS, SECANTS AND CHORDS

• PROBLEM 7-28

Prove that the measure of the angle formed by a tangent and a chord of a circle is one-half the measure of its intercepted arc.

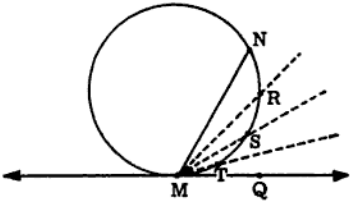


Figure 1

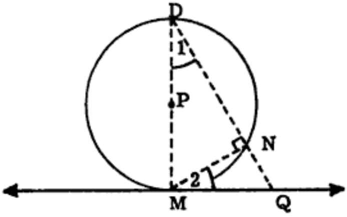


Figure 2

Solution: To see why the statement must be true, consider the inscribed angle $\angle NMR$. We know that $m\angle NMR = \frac{1}{2}\widehat{NR}$. Similarly, for the larger angle $\angle NMS$, it is true that $m\angle NMS = \frac{1}{2}m\widehat{NRS}$. Even as we pick points on the circle closer to M, the statement continues to be true. At the limit, the series of lines $\overleftrightarrow{MR}$, $\overleftrightarrow{MS}$, $\overleftrightarrow{MT}$ ... approach the tangent $\overline{MQ}$, and what we shall prove here is that the relationship still holds:

$m\measuredangle NMQ = \frac{1}{2}m\overset{\frown}{NSM}$.

There are two methods for proving the relation and both involve relating the angle $\measuredangle NMQ$ with some inscribed angle of the circle that intercepts the same arc.

Method 1: Let D be the point on the circle such that $\overline{DM}$ is a diameter. Define Q such that Q is the intersection of $\overleftrightarrow{DN}$ and the tangent. Inscribed angle $\measuredangle 1 = \frac{1}{2}m\overset{\frown}{MN}$. If we prove $\measuredangle 2 \cong \measuredangle 1$, then $\measuredangle 2 = \frac{1}{2}m\overset{\frown}{MN}$ and the proof is complete. We prove $\measuredangle 2 \cong \measuredangle 1$ by showing $\triangle NMQ \sim \triangle MDQ$ by the A-A Similarity Theorem.

Given: $\overleftrightarrow{MQ}$ is tangent to $\odot P$;
$\overline{DM}$ is a diameter of $\odot P$;
$\overline{MN}$ is a chord of $\odot P$.

Prove: $m\measuredangle NMQ = \frac{1}{2}m\overset{\frown}{MN}$

Statement	Reason
1. $\overleftrightarrow{MQ}$ is tangent to $\odot P$; $\overline{DM}$ is a diameter of $\odot P$; $\overline{MN}$ is a chord of $\odot P$.	1. Given.
2. $\overline{PM} \perp \overline{MQ}$.	2. The tangent to a circle is perpendicular to the radius of the circle drawn to the point of intersection.
3. $\measuredangle DMQ$ is a right angle.	3. Definition of perpendicularity.
4. $\triangle DMN$ is a right triangle.	4. If one side of a triangle inscribed in circle is a diameter, then the triangle is a right triangle with the hypotenuse being the diameter.
5. $\measuredangle DNM$ is a right angle.	5. Definition of a right triangle.
6. $\measuredangle MNQ$ is a right angle	6. If one angle of a linear pair is a right angle, then the other angle is also a right angle.
7. $\measuredangle MNQ \cong \measuredangle DMQ$	7. All right angles are congruent.
8. $\measuredangle NQM \cong \measuredangle NQM$	8. A-A Similarity Theorem.
9. $\measuredangle 1 \cong \measuredangle 2$	9. Corresponding angles of similar triangles are congruent.

10. $m\measuredangle 1 = \frac{1}{2}m\overset{\frown}{MN}$	10. The measure of an inscribed angle equals one-half the measure of the intercepted arc.
11. $m\measuredangle 2 = \frac{1}{2}m\overset{\frown}{MN}$	11. Substitution Postulate

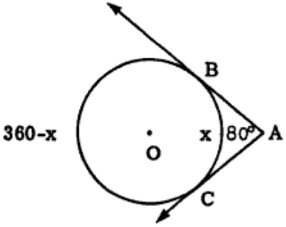
Method 2: Here, again, we show $\measuredangle 1 \cong \measuredangle 2$. By showing $\measuredangle 1$ and $\measuredangle DMN$ are complementary, and $\measuredangle 2$ and $\measuredangle DMN$ are complementary, we can equate $\measuredangle 1$ and $\measuredangle 2$ and, therefore, $m\measuredangle 1 = m\measuredangle 2 = \frac{1}{2}m\overset{\frown}{MN}$.

Statement	Reason
The first 5 steps are the same as the last proof.	
6. $\measuredangle 1$ and $\measuredangle DMN$ are complementary.	6. The acute angles of a right triangle are complementary.
7. $\measuredangle 2$ and $\measuredangle DMN$ are complementary	7. If the sum of two angles is 90°, then they are complementary.
8. $\measuredangle 1 \cong \measuredangle 2$	8. Two angles that are complementary to the same angle are congruent.
9. $m\measuredangle 1 \cong \frac{1}{2}m\overset{\frown}{MN}$	9. The measure of an inscribed angle equals one-half the measure of the intercepted arc.
10. $m\measuredangle 2 = \frac{1}{2}m\overset{\frown}{MN}$	10. Substitution Postulate.

• PROBLEM 7-29

The angle formed by two tangents drawn to a circle from the same external point measures 80°. Find the measure of the minor intercepted arc.

Solution: The measure of an angle formed by two tangents drawn to a circle from an outside point is equal to one-half the difference of the measures of the intercepted arcs.



If we let x = the measure of minor arc $\overset{\frown}{BC}$, then $360°-x$ = the measure of the major arc $\overset{\frown}{BC}$. Applying the above rule,

$$m\measuredangle BAC = \frac{1}{2}(m(\text{major } \overset{\frown}{BC}) - m(\text{minor } \overset{\frown}{BC})).$$

By substitution,

$$80° = \frac{1}{2}(360° - 2x).$$

$80° = 180° - x$, which implies $x = 100°$. Therefore, the measure of the minor intercepted arc is 100°.

An alternative solution for this problem is as follows. Draw $\overline{BC}$. Since tangents to a circle from the same external point are congruent, triangle ABC is an isosceles triangle. Hence, $m\measuredangle ABC = m\measuredangle ACB$. But the sum of the measures of the angles of any triangle is 180°. Therefore,

$$m\measuredangle BAC + m\measuredangle ABC + m\measuredangle ACB = 180°.$$

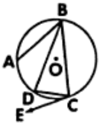
Since $m\measuredangle ABC = m\measuredangle ACB$ and $m\measuredangle BAC = 80°$, we obtain

$$m\measuredangle ABC = m\measuredangle ACB = 50°.$$

Angle ABC is formed by a tangent and a chord. Therefore, its measure equals $\frac{1}{2}$ the measure of the intercepted arc; i.e., $m\measuredangle ABC = \frac{1}{2}m\overset{\frown}{BC}$. By substituting 50° for the $m\measuredangle ABC$, we can calculate the $m\overset{\frown}{BC}$ to be 100°. This method is less direct than the first, but is a valid option.

• PROBLEM 7-30

As shown in the diagram, $\overline{BD}$ bisects $\measuredangle ABC$, which is inscribed in circle O. $\overrightarrow{EC}$ is tangent to circle O at point C. Prove that $\measuredangle ABD \cong \measuredangle DCE$.

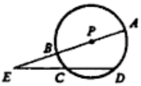


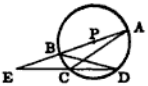
Solution: This proof will involve recalling that both the measure of an inscribed angle in a circle and the measure of an angle formed by a tangent and a chord to that tangency point are equal to one-half the measure of the intercepted arc. With this in mind, we can prove $\measuredangle DCE \cong \measuredangle DBC$. By applying the definition of an angle bisector and the transitive property of congruence, we can conclude that $\measuredangle ABD = \measuredangle DCE$.

Statement	Reason
1. $\overline{BD}$ bisects $\sphericalangle ABC$	1. Given.
2. $\sphericalangle ABD \cong \sphericalangle DBC$	2. A bisector divides an angle into two congruent angles.
3. $\overleftrightarrow{EC}$ is tangent to the circle.	3. Given.
4. $m\sphericalangle DCE = \frac{1}{2}m\overset{\frown}{DC}$	4. The measure of an angle formed by a tangent and a chord at the point of tangency is equal to one-half the measure of the intercepted arc.
5. $m\sphericalangle DBC = \frac{1}{2}m\overset{\frown}{DC}$	5. The measure of an inscribed angle is equal to one-half the measure of the intercepted arc.
6. $m\sphericalangle DBC = m\sphericalangle DCE$	6. Transitive property of equality.
7. $\sphericalangle DBC \cong \sphericalangle DCE$	7. Two angles are congruent if their measures are equal.
8. $\sphericalangle ABD \cong \sphericalangle DCE$	8. Transitive property of congruence.

• PROBLEM 7-31

Two secant lines of the same circle share an endpoint in the exterior of the circle. Show that the product of the lengths of one secant segment and its external segment equal the product of the lengths of the other secant segment and its external segment.





Solution: The secant segment is the portion of the secant line bounded by the common endpoint and the point of intersection with the circle farthest from this point.

Restating the problem in terms of the above figure, we must show that $EB \cdot EA = EC \cdot ED$. We can rewrite this as the proportion $\frac{EB}{ED} = \frac{EC}{EA}$. To prove the proportion, we show $\Delta EBD \sim \Delta ECA$ by the A-A Similarity Theorem.

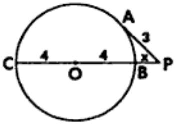
Given: $\overline{ABE}$ and $\overline{DCE}$ are secant segments of $\odot P$. Point E is

exterior to OP.
Prove: EB · EA = EC · ED.

Statements	Reasons
1. $\overline{ABE}$ and $\overline{DCE}$ are secant segments of ⊙P. Point E is exterior to ⊙P.	1. Given.
2. $m\measuredangle BAC = \frac{1}{2}m\overset{\frown}{BC}$ $m\measuredangle BDC = \frac{1}{2}m\overset{\frown}{BC}$	2. The measure of an inscribed angle equals one-half the measure of the subtended arc.
3. $m\measuredangle BAC = m\measuredangle BDC$	3. Transitivity Postulate.
4. $\measuredangle BAC \cong \measuredangle BDC$	4. Definition of congruence of angles.
5. $\measuredangle E \cong \measuredangle E$	5. An angle is congruent to itself.
6. $\Delta EBD \sim \Delta ECA$	6. The A-A Similarity Theorem.
7. $\frac{EB}{ED} = \frac{EC}{EA}$	7. The sides of similar triangles are proportional.
8. EB · EA = EC · ED.	8. In a proportion, product of the means must equal the product of the extremes.

• PROBLEM 7-32

In the accompanying figure, the length of a radius of circle O is 4. From a point P outside circle O, tangent $\overline{PA}$ is drawn. If the length of $\overline{PA}$ is 3, find the distance from P to the circle.



Solution: Let B be the point of intersection of the circle with the segment connecting point P to the center of ⊙O. $\overline{PB}$ is the shortest segment that can be drawn from P to any point on the circle. Therefore, PB is the distance from P to the circle.

Now locate point C such that $\overline{PC}$ intersects the circle at B and C and passes through both O and P.

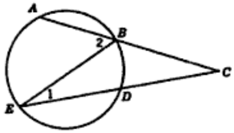
From this, we can determine that $\overline{PC}$ is a secant to circle O and, as such, the tangent $\overline{PA}$ will be, by theorem, the mean proportional between the length of the secant, $\overline{PC}$, and the length of its external segment, $\overline{PB}$. Thus, $PC:PA = PA:PB$ is the proportion.

Diameter $CB = 8$, since we are given the radius of the circle as 4.

If we let x = the length of $\overline{PB}$, then $x + 8$ = the length of $\overline{PC}$. By substitution, $(x+8):3 = 3:x$. As stated in prior examples, the product of the means equals the product of the extremes. Therefore, $x(x+8) = 9$. By the distributive property, this becomes $x^2 + 8x = 9$ or $x^2 + 8x - 9 = 0$. To solve for x, factor the left side of the equation and set each factor equal to zero. $(x-1)$ and $(x+9)$ are the factors. $x-1 = 0$ implies $x = 1$ and $x+9 = 0$ implies $x = -9$. The negative value has no geometric significance and is rejected. Therefore, the distance from P to the circle, PB, equals 1.

• PROBLEM 7-33

In the accompanying figure, we are given that $m\overset{\frown}{AE} = 80°$ and $m\measuredangle C = 20°$. Find the measures of $\measuredangle 1$, $\measuredangle 2$ and $\overset{\frown}{BD}$.



Solution: $\measuredangle 2$ is an inscribed angle which intercepts an arc, $\overset{\frown}{AE}$, of 80°. Recall that the measure of an inscribed angle is equal to one-half the measure of its intercepted arc. Hence, $\measuredangle 2 = 40$.

We can derive $m\overset{\frown}{BD}$ by noticing that it is intercepted by $\measuredangle C$. $\measuredangle C$ is formed by the intersection of two secants outside the circle. Recall that an angle formed in this way is equal to one-half the difference of its intercepted arcs. The intercepted arcs of $\measuredangle C$ are $\overset{\frown}{AE}$ and $\overset{\frown}{BD}$. Therefore, $\measuredangle C = \frac{1}{2}(\overset{\frown}{AE} - \overset{\frown}{BD})$. By substituting the given values for $\measuredangle C$ and $\overset{\frown}{AE}$, we can solve for $m\overset{\frown}{BD}$.

$$20° = \frac{1}{2}(80° - \overset{\frown}{BD})$$

$$40° = 80° - \overset{\frown}{BD}$$

$$\overset{\frown}{BD} = 40°.$$

Since $\measuredangle 1$ is an inscribed angle which intercepts this 40° arc, $\measuredangle 1$ must measure one-half of 40°, or 20°.

Therefore, $m\measuredangle 1 = 20°$, $m\measuredangle 2 = 40°$, and $m\overset{\frown}{BD} = 40°$.

3. $m\angle M + m\angle MBC + m\angle MCB = 180°$.	3. The angle sum of a triangle equals 180°.
4. $m\angle MBC + m\angle MCB = 180 - m\angle M$.	4. Subtraction Property of Equality.
5. $m\angle NBC + m\angle MBC = 180$ $m\angle BCP + m\angle MCB = 180$	5. Pairs of angles that form a linear pair sum to 180°.
6. $m\angle NBC + m\angle BCP + m\angle MBC + m\angle MCB = 360$	6. Addition Property of Equality.
7. $m\angle NBC + m\angle BCP = 360 - (m\angle MBC + m\angle MCB) = 360 - (180 - m\angle M) = 180 + m\angle M$.	7. The Subtraction Property of Equality. The property that the sum of the interior angles of a triangle equals 180, together with the substitution postulate.
8. $\angle ONB$ is a right angle $\angle OPC$ is a right angle	8. The intersection of a tangent with a radius at point of tangency forms a right angle.
9. $\overline{OB} \cong \overline{OB}$	9. Reflexive property.
10. $\overline{OB}$ is a hypotenuse	10. A sides of a triangle opposite a right angle is the hypotenuse.
11. $\overline{ON} \cong \overline{OP} \cong \overline{OA}$	11. All radii of a circle are equal.
12. $\triangle ONB \cong \triangle OAB$	12. The Hypotenuse-Leg Theorem.
13. $\triangle OAC \cong \triangle OPC$	13. The Hypotenuse-Leg Theorem.
14. $\angle NBO \cong \angle ABO$ ($\overline{OB}$ bisects $\angle AON$ $\angle NBA$) $\angle OCA \cong \angle OCP$ ($\overline{OC}$ bisects $\angle AOP$ $\angle ACP$)	14. Corresponding parts of congruent triangles are congruent, and definition of angle bisector.
15. $m\angle NBO = m\angle ABO = \frac{1}{2}m\angle NBC$. $m\angle OCA = m\angle OCP = \frac{1}{2}m\angle BCP$.	15. The Angle Addition Postulate,
16. $m\angle ABO + m\angle OCA = \frac{1}{2}(m\angle NBC + m\angle BCP) = \frac{1}{2}(180 + m\angle M)$.	16. Angle Addition Postulate, and Substitution Postulate from step 7.
17. $m\angle BOC + m\angle ABO + m\angle OCA = 180$.	17. The angle sum of a triangle is 180°.

adjacent vertices. The side we choose is $\overline{BD}$. The angles we must show congruent are $\measuredangle A$ ($\measuredangle BAD$) and $\measuredangle E$ ($\measuredangle BED$).

To show $\measuredangle A \cong \measuredangle E$, we relate the triangles $\triangle AFD$ and $\triangle EFB$. Note: Since the angle sum of a triangle is 180°, $m\measuredangle A + m\measuredangle AFD + m\measuredangle FDA = 180°$ and $m\measuredangle E + m\measuredangle EBF + m\measuredangle EFB = 180°$. Thus,

(1) $m\measuredangle A + m\measuredangle AFD + m\measuredangle FDA = m\measuredangle E + m\measuredangle EBF + m\measuredangle EFB$. But $\measuredangle AFD$ and $\measuredangle EFB$ are opposite angles. Thus, $m\measuredangle AFD = m\measuredangle EFB$, and we delete them from the equation.

(2) $m\measuredangle A + m\measuredangle FDA = m\measuredangle E + m\measuredangle EBF$. We are given that $m\measuredangle A = \frac{1}{2}m\measuredangle ABE = \frac{1}{2}m\measuredangle FBE$ or $m\measuredangle FBE = 2 \cdot m\measuredangle A$

(3) $m\measuredangle A + m\measuredangle FDA = m\measuredangle E + 2m\measuredangle A$.

We have an equation relating $\measuredangle A$, $\measuredangle E$, and the unwanted $\measuredangle FDA$. We rid ourselves of $\measuredangle FDA$ by showing $m\measuredangle FDA = 2 \cdot m\measuredangle E$. Note that $\measuredangle FDA$ is an exterior angle of $\triangle EDC$. Its measure equals the sum of remote interior angles $\measuredangle E$ and $\measuredangle C$. But $\triangle EDC$ is isosceles. Thus, $\measuredangle E \cong \measuredangle C$. Since $m\measuredangle FDA = m\measuredangle E + m\measuredangle C$, $m\measuredangle FDA = 2 \cdot m\measuredangle E$. Using this result in (3), we have:

(4) $m\measuredangle A + 2 \cdot m\measuredangle E = m\measuredangle E + 2 \cdot m\measuredangle A$

(5) $m\measuredangle A = m\measuredangle E$ or $\measuredangle A = \measuredangle E$.

Since side $\overline{DB}$ intercepts congruent angles from the non-adjacent vertices, quadrilateral ABDE is cyclic.

Statements	Reasons
1. (See the problem statement.)	1. Given.
2. $\measuredangle E \cong \measuredangle BCD$	2. The base angles of an iso celes triangle are congruent.
3. $m\measuredangle ADE = m\measuredangle E + m\measuredangle DCE$ $= 2 \cdot m\measuredangle E$.	3. The measure of an exterior angle of the triangle equals the sum of the measures of the remote interior angles.
4. $m\measuredangle AFD \cong m\measuredangle EFB$.	4. Opposite angles formed by intersecting lines are congruent.
5. $m\measuredangle AFD + m\measuredangle A + m\measuredangle ADF = 180$ $m\measuredangle E + m\measuredangle EFB + m\measuredangle FBE = 180$	5. The sum of the angles of a triangle equals 180°.
6. $m\measuredangle AFD + m\measuredangle A + 2 \cdot m\measuredangle E = 180$ $m\measuredangle AFD + m\measuredangle E + 2 \cdot m\measuredangle A = 180$.	6. Substitution Postulate. (Steps 5, 4, 3, and 1).
7. $m\measuredangle E - m\measuredangle A = 0$.	7. Subtraction Postulate.

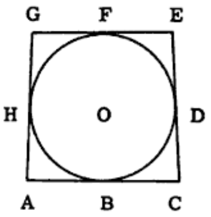
5. ∢DAB and ∢QBD are complementary.	5. The acute angles of right triangle ADB are complementary.
6. ∢AQC and ∢BQC are right angles.	6. Perpendicular lines intersect at right angles.
7. ∢ACQ and ∢DAB are complementary.	7. The acute angles of right triangles △CAQ are complementary.
8. ∢ACQ ≅ ∢QBD.	8. Angles complementary to congruent angles are congruent.
9. $\overline{CQ} \cong \overline{CQ}$.	9. A segment is congruent to itself.
10. $\overline{AQ} \cong \overline{QB}$.	10. The radii of a circle are congruent.
11. ∢AQC ≅ ∢BQC.	11. All right angles are congruent.
12. △AQC ≅ △BQC.	12. The SAS Postulate.
13. ∢ACQ ≅ ∢BCQ.	13. Corresponding parts of congruent triangles are congruent.
14. ∢BCQ ≅ ∢QBD.	14. Substitution Postulate. (Steps 13, 8).
15. ∢BCQ ≅ ∢QDB.	15. Substitution Postulate. (Steps 14, 3)
16. Quadrilateral CDQB is cyclic.	16. If a side of a quadrilateral subtends equal angles with the two nonadjacent vertices, then the quadrilateral is cyclic.

CIRCLES INSCRIBED IN POLYGONS

• PROBLEM 7-39

Show that in a quadrilateral circumscribed about a circle, the sum of the lengths of a pair of opposite sides equals the sum of the lengths of the remaining pair of opposite sides.

Solution: In the accompanying figure, quadrilateral AGEC is circumscribed about ⊙O and is tangent to ⊙O at points H, F, D, and B. We are asked to show



(i) GE + AC = AG + CE.

We can accomplish this by rewriting each side of the equation in terms of segments such that each segment length on the left side of the equation has a segment equal in length to it on the opposide side.

To find these congruent segments, we use the result that the two tangent segments to a circle from an external point are congruent. Consider a vertex G and the two segments drawn to the circle, $\overline{GF}$ and $\overline{GH}$. Since $\overline{GH}$ is tangent to ⊙O at H and $\overline{GF}$ is tangent to ⊙O at F, $\overline{GF}$ and $\overline{GH}$ are tangent segments drawn from external point G and, thus, $\overline{GF} = \overline{GH}$. If we consider E, C, and A as external points, then we relate the tangent segments by the congruencies $\overline{EF} \cong \overline{ED}$, $\overline{CD} \cong \overline{CB}$, and $\overline{AB} \cong \overline{AH}$.

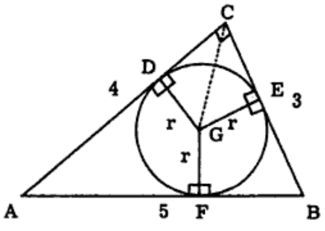
We can rewrite the equation to be proved in terms of the tangent segments. Note that GE = GF + FE, AC = AB + BC, GA = GH + HA, and EC = ED + DC. Substituting these results in (i), we obtain:

(ii) GF + FE + AB + BC = GH + HA + ED + DC.

Since GF = GH, FE = ED, AH = AB, and CB = CD, the equation holds and, thus, GE + AC = AG + CE.

• PROBLEM 7-40

Find the radius of a circle inscribed in a triangle whose sides have lengths 3, 4 and 5.



Solution: In the accompanying figure, ΔABC circumscribes circle G. AC = 4, BC = 3, and AB = 5. Points D, E, and F are the points of tangency of ⊙G with the sides of ΔABC. We are asked to find GE - the length of the radius of the inscribed circle. We will do this by (1) finding the length of tangent segment, CE, and (2) showing ΔCEG is isosceles with $\overline{GE} \cong$ tangent segment $\overline{CE}$. Thus, the radius, r, will equal CE.

(1) To find the lengths of the tangent segments, we use the theorem which states that, from an external point (such as C), the tangent segments (DC and CE) drawn to the circle (G) are congruent ($\overline{DC} \cong \overline{CE}$). Thus, if we consider C, A, and B respectively as the external point, we obtain:

(i) DC = CE (C is the external point)
(ii) AD = AF (A is the external point)
(iii) BF = BE (B is the external point)

But from the figure, we know that

(iv) AD + DC = AC = 4
(v) CE + EB = CB = 3
(iv) AF + FB = AB = 5.

We have 6 equations with 6 variables. To solve for tangent segment CE, we first eliminate AD, BE, and DC by substituting in equations (iv), (v), and (vi) the values AD = AF, BE = BF, and DC = CE.

(vii) AF + CE = 4
(viii) CE + BF = 3
(ix) AF + FB = 5.

We now have 3 equations in 3 unknowns. Eliminate AF by substituting (ix) in (vii).

(x) CE - FB = 4 - 5 = -1
(xi) CE + FB = 3

We eliminate FB by adding (x) and (xi).

(xii) 2 · CE = 2 or CE = 1.

(2) Thus, we have a value for CE.

To show ΔCEG is isosceles, we will show m∢ECG = m∢EGC = 45°. In order to show m∢ECG = 45°, we prove ∢DCG ≅ ∢ECG. By the Hypotenuse Leg Theorem, ΔDCG ≅ ΔECG, since points D and E are points of tangency, radii $\overline{DG} \perp \overline{DC}$ and $\overline{GE} \perp \overline{CE}$ and, thus, m∢CDG = m∢CEG = 90°. Since $\overline{DG}$ and $\overline{GE}$ are radii, $\overline{DG} \cong \overline{GE}$ and $\overline{CG} \cong \overline{CG}$. Thus, ∢DCG ≅ ∢ECG because they are corresponding angles of congruent triangles. Because a 3-4-5 triangle is a right triangle, ∢ACB is a right angle. Thus, we have

m∢DCG + m∢ECG = m∢ACB.

By substitution, 2·m∢ECG = 90°
m∢ECG = 45°.

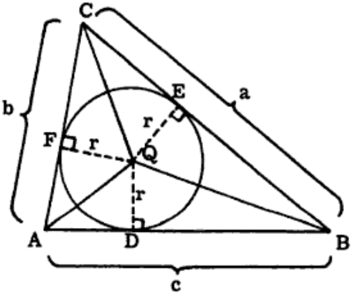
We will now show $m\angle CGE$ also equals 45°. Since $m\angle ECG = 45°$ and $m\angle GEC = 90°$, then to find $m\angle CGE$, we have the following:

$$\begin{aligned} m\angle GCE + m\angle GEC + m\angle CGE &= 180° \\ 45° + 90° + m\angle CGE &= 180° \\ m\angle CGE &= 45°. \end{aligned}$$

Since $m\angle CGE \cong m\angle GCE$, ΔCGE is isosceles. Therefore, $GE \cong CE$, and $r = GE = CE = 1$. The radius of the inscribed circle equals 1.

• PROBLEM 7-41

Given: ΔABC with sides of length a, b, and c. The inscribed circle, Q, intersects $\overline{AB}$ at D, $\overline{BC}$ at E, and $\overline{CA}$ at F. Find the lengths AD and DB in terms of a, b, and c.



Solution: We are asked to find AD and DB, the lengths of the segments into which the tangency point, D, divides side AB of ΔABC, in terms of the sides of the triangle. We know therefore, that we must (1) use the properties of the inscribed circle Q; and (2) relate segments $\overline{AD}$ and $\overline{DB}$ to the other sides of ΔABC.

Because D, E, and F are the points of tangency, the radii $\overline{QD}$, $\overline{QE}$, and $\overline{QF}$ are all radii to the points of tangency and, therefore, perpendicular to the sides.

Thus, from the fact that $\odot Q$ is the inscribed circle, we know that (1) $QD = QE = QF$ (because all radii are congruent); and (2) $\overline{QD} \perp \overline{AB}$, $\overline{QE} \perp \overline{CB}$, and $\overline{QF} \perp \overline{AC}$.

We now relate segments $\overline{AD}$ and $\overline{DB}$ to the corresponding sides. Note, from the problem statement, that $\overline{AD}$ and $\overline{DB}$ are segments divided by the point of tangency. We can use the same procedure as we are using now to find AF and FC, or CE and EB. Therefore, it should not be surprising that the expressions for these segments be, in some way, similar to the expressions for AD and DB. Nor should it be unexpected that, when we relate segments AD and DB to side AC, those segments correspond to segments AF and FC. In fact, we can relate segment $\overline{AD}$ to $\overline{AF}$, they are congruent. Consider $\triangle AQD$ and $\triangle AQF$. $\overline{QF} = \overline{QD}$ because they are radii of $\odot Q$. $\angle QFA$ and $\angle QDA$ are both right angles and therefore congruent. $\overline{AQ} \cong \overline{AQ}$ is true by the reflexive property. Since the hypotenuse and the leg of right triangle $\triangle AQD$ are congruent to the hypotenuse and the leg of right triangle $\triangle AQF$, $\triangle AQD \cong \triangle AQF$. By corresponding parts, $\overline{AD} \cong \overline{AF}$.

In relating segments $\overline{AD}$ and $\overline{DB}$ to side $\overline{CB}$, we note that $\triangle QDB \cong \triangle QEB$ and, therefore, $\overline{DB} \cong \overline{EB}$. Segments CF and CE cannot be related to $\overline{AD}$ and $\overline{DB}$, but they can be related to each other. By showing $\triangle QFC \cong \triangle QEC$, we know by corresponding parts that $\overline{CF} \cong \overline{CE}$.

We have (1) used the properties of inscribed circle Q and (2) related $\overline{AD}$ and $\overline{DB}$ to the sides of $\triangle ABC$. We collect our data, and solve for AD and DB.

Because F, E, and D are points on $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$, respectively, we have from the definition of "betweenness" that:

$$\begin{aligned} AB &= AD + DB \\ AC &= AF + FC \\ BC &= BE + EC. \end{aligned}$$

From the properties of the inscribed circles we have shown that

$$\begin{aligned} \overline{AD} &\cong \overline{AF} \\ \overline{DB} &\cong \overline{EB} \\ \overline{FC} &\cong \overline{EC} \end{aligned}$$

From the given, we know that

$$\begin{aligned} BC &= a \\ AC &= b \\ AB &= c. \end{aligned}$$

We substitute a, b, c for BC, AC, and AB and substitute AD, DB, and FC for AF, EB, and EC in the above "betweenness" equations to obtain the following:

$$\begin{aligned} c &= AD + DB \\ b &= AD + FC \\ a &= DB + FC \end{aligned}$$

Here, we have three equations in three unknowns. FC is a variable for which we do not wish to solve. Therefore, we eliminate that variable first. Since $b = AD + FC$, $FC = b - AD$. Substituting this result in the equation, we obtain the following:

$$\begin{aligned} c &= AD + DB \\ a &= DB + (b-AD) = -AD + DB + b \end{aligned}$$

Adding the equations, we eliminate AD.

$$c + a = (AD-AD) + DB + DB + b = 2DB + b$$

$$DB = \frac{c + a - b}{2}$$

To solve for AD, remember that $c = AD + DB$ and, therefore, $AD = c - DB$.

$$AD = c - \frac{c + a - b}{2} = \frac{2c - (c+a-b)}{2} = \frac{c + b - a}{2}$$

Remember the earlier supposition that the expressions would probably be similar? Segment $\overline{DB}$ touches vertex B, and its formula can be written as

$$\frac{\left\{\begin{matrix}\text{sum of the length of the}\\ \text{sides of } \Delta ABC \text{ that touch B}\end{matrix}\right\} - \left\{\begin{matrix}\text{length of the}\\ \text{third side}\end{matrix}\right\}}{2}.$$

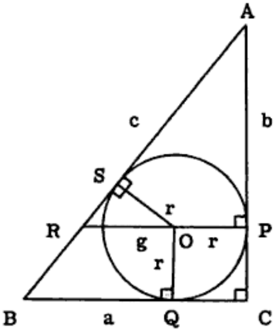
The same formula can be written for AD. For any segment that touches an arbitrary vertex X, the formula for the length is

$$\frac{\left\{\begin{matrix}\text{sum of the lengths of}\\ \text{sides that touch x}\end{matrix}\right\} - \left\{\begin{matrix}\text{length of the}\\ \text{third side}\end{matrix}\right\}}{2}$$

• PROBLEM 7-42

Find the radius, r, of the inscribed circle of right triangle ABC in terms of leg lengths a and b and hypotenuse length c.

Solution: In the accompanying figure, circle O of radius r is inscribed in right ΔABC with $AB = c$, $BC = a$, and $AC = b$. Points P, Q, and S are points of tangency of the circle with the triangle. Point R is the intersection of $\overline{AB}$ and the extension of $\overline{OP}$. g is the length $\overline{OR}$. We must find r in terms of a, b, and c.



First, we will show $\Delta ARP \sim \Delta ABC$, which will give a relationship involving (known) a, b, and c and (unknown) g and

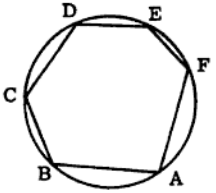
Solving for r, we obtain

$$\begin{aligned} &\text{(vii)} && rc = ab-ar-rb \\ &\text{(viii)} && ra+rb+rc = ab \\ &\text{(ix)} && r(a+b+c) = ab \\ &\text{(x)} && r = \frac{ab}{a+b+c} \end{aligned}$$

CIRCLES CIRCUMSCRIBING POLYGONS

• PROBLEM 7-43

Prove that the sum of the alternate angles of any hexagon inscribed in a circle is equal to four right angles.



Solution: We see from the figure that each angle is an inscribed angle of the circle. As such, each angle has a measure equal to $\frac{1}{2}$ the measure of its intercepted arc. If the alternate angles of the hexagon sum to 360°, the measure of four right angles, then we have proved the desired results.

We choose $\angle A$, $\angle E$, and $\angle C$ as the alternate angles.

$$m\angle A = \frac{1}{2}m\overset{\frown}{BCDEF}$$

$$m\angle E = \frac{1}{2}m\overset{\frown}{FABCD}$$

$$m\angle C = \frac{1}{2}m\overset{\frown}{DEFAB}$$

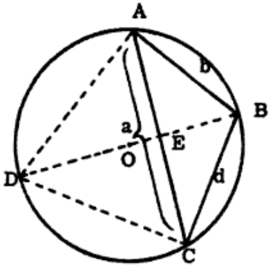
Hence, $m\angle A + m\angle E + m\angle C = \frac{1}{2}(m\overset{\frown}{BCDEF} + m\overset{\frown}{FABCD} + m\overset{\frown}{DEFAB})$

We now show that these three arcs form two entire circular angles. Note:

$$\begin{aligned} m\overset{\frown}{BCDEF} &= m\overset{\frown}{BC} + m\overset{\frown}{CD} + m\overset{\frown}{DE} + m\overset{\frown}{EF} \\ m\overset{\frown}{FABCD} &= m\overset{\frown}{FA} + m\overset{\frown}{AB} + m\overset{\frown}{BC} + m\overset{\frown}{CD} \\ m\overset{\frown}{DEFAB} &= m\overset{\frown}{DE} + m\overset{\frown}{EF} + m\overset{\frown}{FA} + m\overset{\frown}{AB} \end{aligned}$$

Adding these three equations, we obtain

3. $\angle F \cong \angle C$	3. The interior angles of a regular polygon are congruent.
4. $\Delta AFE \cong \Delta BCD$	4. The SAS Postulate.
5. $\overline{AE} \cong \overline{BD}$	5. Corresponding parts of congruent triangles are congruent.
6. Quadrilateral ABDE is a parallelogram	6. If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallogram.
7. G is the center of ABCDEF	7. Every regular polygon has a center that is concurrent with the centers of the inscribed and circumscribed circles.
8. $\overline{AG} \cong \overline{BG} \cong \overline{CG} \cong \overline{DG} \cong \overline{EG} \cong \overline{FG}$	8. The radii of a regular polygon are congruent.
9. $\Delta AGB \cong \Delta AGF \cong \Delta FGE \cong \Delta EGD \cong \Delta DGC \cong \Delta CGB$	9. The SSS Postulate.
10. $\angle AGF \cong \angle FGE \cong \angle EGD \cong \angle DGC \cong \angle CGB \cong \angle BGA$	10. Corresponding parts of congruent triangles are congruent.
11. $m\angle AGF + m\angle FGE + m\angle EGD + m\angle DGC + m\angle CGB + m\angle BGA = 360°$	11. Angles that form a complete circle sum to 360°.
12. $6 \cdot m\angle AGF = 360°$	12. Substitution Postulate.
13. $m\angle AGF = 60°$	13. Division Property of Equality.
14. $m\angle AGD = m\angle AGF + m\angle FGE + m\angle EGD$ $m\angle BGE = m\angle BGA + m\angle AGF + m\angle FGE$	14. Angle Addition Postulate.
15. $m\angle AGD = 60° + 60° + 60° = 180°$ $m\angle BGE = 60° + 60° + 60° = 180°$	15. Substitution Postulate.
16. $\angle AGD$ and $\angle BGE$ are straight angles and $\overline{AGD}$ and $\overline{BGE}$	16. A straight angle has measure 180°.
17. $AD = AG + DG = 2 \cdot AG$ $BE = BG + EG = 2 \cdot AG$	17. Definition of "Betweenness" and Substitution



Consider the quadrilateral determined by diameter $\overline{DC}$, cyclic quadrilateral ABCD. Side $\overline{AB}$ has known length b. Diagonal $\overline{AC}$ has known length a. Diagonal $\overline{DB}$ is a diameter and has known length 2 · radius or 2. Furthermore, because DB is a diameter, ΔDAB and ΔDCB are triangles inscribed in a semicircle and therefore are right triangles. Side $\overline{DA}$ is thus a leg of right ΔDAB and can be expressed in terms of known quantities using the Pythagorean Theorem. $DA^2 = DB^2 - AB^2$ or, substituting the given, $DA^2 = 2^2 - b^2$ or $DA = \sqrt{4-b^2}$.

Of the six elements (4 sides, 2 diagonals) used in Ptolemy's Theorem, we have solved for 4, one element is the unknown, and DC is an element that is also unknown but for which we are not solving. We recap our information up to this point:

$AB = b,\ AC = a,\ DB = 2,\ AD = \sqrt{4-b^2}$.

(i) By Ptolemy's Theorem, $DB \times AC = AD \times BC + AB \times DC$.

(ii) Because of rt. ΔDCB, $DC^2 + BC^2 = DB^2$.

Substituting our known values in (i) and (ii), we obtain:

(iii) $2 \cdot a = \sqrt{4-b^2} \cdot BC + b \cdot DC$

(iv) $DC^2 + BC^2 = 4$.

We then solve (iv) for DC in terms of BC. Then we substitute for DC in (iii) to obtain an expression totally in BC: From (iv), we have $DC = \sqrt{4-BC^2}$.

(v) $2 \cdot a = \sqrt{4-b^2} \cdot BC + b\sqrt{4-BC^2}$.

We now have an expression for BC, but we have a long algebraic road between (v) and a solution for BC. First, we eliminate all BC's in radicals by subtracting $\sqrt{4-b^2} \cdot BC$ from both sides and squaring.

(vi) $(2a - \sqrt{4-b^2}BC)^2 = (\sqrt{4-b^2}BC + b\sqrt{4-BC^2} - \sqrt{4-b^2}BC)^2$

(vii) $(2a - \sqrt{4-b^2} \cdot BC)^2 = (b\sqrt{4-BC^2})^2$

(viii) $4a^2 - 4a \cdot BC \cdot \sqrt{4-b^2} + 4BC^2 - BC^2 \cdot b^2 = 4b^2 - b^2BC^2$

Gathering terms on the left side, we have:

(ix) $4BC^2 - (4a\sqrt{4-b^2})BC + (4a^2 - 4b^2) = 0$

We now divide each side by 4 and use the quadratic formula.

(x) $BC^2 + (-a\sqrt{4-b^2})BC + (a^2-b^2) = 0.$

$qx^2 + px + r = 0$, $x = \frac{-p\pm\sqrt{p^2-4qr}}{2q}$. In this equation (ix), $q = 1$; $p = -a\sqrt{4-b^2}$; $r = (a^2-b^2)$.

(xi) $BC = \frac{+a\sqrt{4-b^2} \pm \sqrt{(4a^2-a^2b^2) - 4(a^2-b^2)}}{2}$

(xii) $BC = \frac{a\sqrt{4-b^2} \pm \sqrt{4b^2 - a^2b^2}}{2}$

(xiii) $BC = \frac{a}{2}\sqrt{4-b^2} \pm \frac{b}{2}\sqrt{4-a^2}$.

There would seem to be two answers, $BC = \frac{a}{2}\sqrt{4-b^2} + \frac{b}{2}\sqrt{4-a^2}$ and $BC = \frac{a}{2}\sqrt{4-b^2} - \frac{b}{2}\sqrt{4-a^2}$. But $\frac{a}{2}\sqrt{4-b^2} + \frac{b}{2}\sqrt{4-a^2}$ equals the sum of arcs a and b and therefore cannot be their difference. (Consult the previous problem.) Therefore the difference

$$d = \frac{a}{2}\sqrt{4-b^2} - \frac{b}{2}\sqrt{4-a^2}.$$

Note: To see where the extra answer crept in, remember that in step (vi), we squared each side, a procedure that sometimes leads to an extra root in the equation.

CHAPTER 8

AREAS OF PLANE REGIONS

SQUARES AND RECTANGLES

• PROBLEM 8-1

Find the area of a square whose perimeter is 20 ft.

Solution: The perimeter, p, of a square is given by 4 times the length of one side, s; that is, $p = 4s$.

But, by substitution,

$$p = 20 \text{ ft.}$$
$$20 \text{ ft.} = 4s$$
$$5 \text{ ft.} = s \quad .$$

The area of a square is given by s^2 . Hence, $A = s^2$ and, by substitution,

$$A = (5 \text{ ft.})^2 = 25 \text{ ft.}^2$$

Therefore, the area of the square is 25 sq. ft.

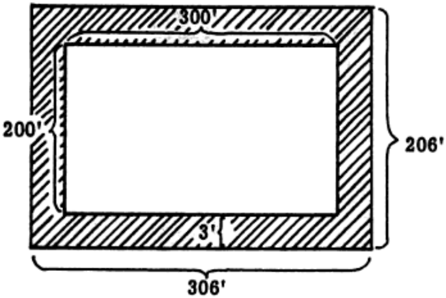
• PROBLEM 8-2

A man has a rectangular piece of property measuring 200 x 300 ft. He plans to put a concrete sidewalk 3 ft. wide around the edge. What will the area of the sidewalk be?

Solution: We assume that the man puts the sidewalk down as shown in the figure. We must find the area of the shaded region. First, find the area of the figure bounded by the outer border (the large rectangle). Next, find the area of the small rectangle and subtract this from the area of the large rectangle. We will then have the area of the sidewalk.

Area of large rectangle = (206 ft.)(306 ft.)
= 63036 sq. ft.

Area of small rectangle = (300 ft.)(200 ft.)
= 60000 sq. ft.



Area of sidewalk = (63036 - 60000) sq. ft.
= 3036 sq. ft.

This is much more efficient than breaking up the sidewalk region into rectangles, calculating their areas, and summing them up.

• PROBLEM 8-3

A man wishes to put linoleum tile on a rectangular floor which measures 5 x 7 yd. If each piece of linoleum tile is a 9 in. square, how many pieces of tile will be needed? If a quart of tile adhesive covers 10 sq. ft., how many quarts will he need?

Solution: The actual problem is to find out how many tiles (9" per side) are needed to cover the surface of a rectangle 5 x 7 yd.

The area of the rectangle (the floor) is made up of the sum of the areas of the square tiles. Let us represent the number of tiles needed by x. Then the previous statement may be written, mathematically, as

$$x(\text{Area of 1 tile}) = \text{Area of floor}. \tag{1}$$

The area of 1 tile is $(9 \text{ in.})^2 = 81$ sq. in., since each tile is a square. The area of the floor is (5 yd.)(7 yd.) = 35 sq. yd., since the floor is rectangular. Using these results in (1), we obtain

$$x(81 \text{ sq. in.}) = 35 \text{ sq. yd.}$$

Dividing both sides by 81 sq. in.,

$$x = \frac{35 \text{ sq. yd.}}{81 \text{ sq. in.}} \tag{2}$$

Now, we have mixed units in (2), In order to get a pure number in (2) (which we must, since the number of tiles (x) is dimensionless) we must change either sq. yds. to sq. ins., or vice versa. We take the former course.

$$1 \text{ sq. yd. } (1 \text{ yd.})^2 = (36 \text{ in.})^2 = 1296 \text{ sq. in.}$$

Using this in (2),

$$x = \frac{(35 \text{ sq. yd.})(1296 \text{ sq. in./sq. yd.})}{(81 \text{ sq. in.})} = 560 \quad .$$

The man will need 560 tiles to cover his floor.

As for the adhesive, we know that each quart of adhesive covers

10 sq. ft. We must cover the area of the floor, which is, from the above calculations, 35 sq. yd. Since

$$1 \text{ sq. yd.} = (1 \text{ yd.})^2 = (3 \text{ ft.})^2 = 9 \text{ sq. ft.}$$

the floor area is $(35)(9)$ sq. ft. $= 315$ sq. ft.

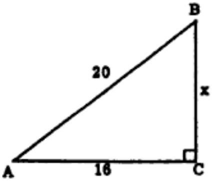
The man will therefore need

$$\frac{315}{10} = 31.5 \text{ quarts of adhesive.}$$

TRIANGLES

• PROBLEM 8-4

In right triangle ABC, ∡ C measures 90°, $\overline{AB}$ is of length 20 in., and the length of $\overline{AC}$ is 16 in. Find the area of triangle ABC.



Solution: Assume, as in the accompanying figure, that $\overline{AC}$ is the base of triangle ABC. Since ∡ C is a right angle, $\overline{BC}$ is the altitude to the base of the triangle.

The area (A) of a right triangle is given by one-half the product of the length of the base (b) times the length of the corresponding altitude (h). $A = \frac{1}{2}bh$.

We are not given the length of the altitude, but can calculate it by applying the Pythagorean Theorem to ΔABC. If we let $x =$ the length of the altitude, then,

$$x^2 + (16 \text{ in.})^2 = (20 \text{ in,})^2$$

$$x^2 + 256 \text{ in.}^2 = 400 \text{ in.}^2$$

$$x^2 = 144 \text{ in.}^2$$

which implies $x = 12$ in.

The area of $\Delta ABC = \frac{1}{2}bh$. By substitution,

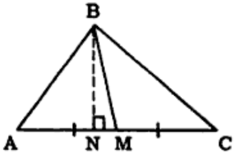
$$\text{area of } \Delta ABC = \tfrac{1}{2}(16 \text{ in.})(12 \text{ in.}) = 96 \text{ in.}^2$$

Therefore, the area of ΔABC is 96 sq. in.

• PROBLEM 8-5

The area of a certain triangle is 52 square feet and the height is 13 feet. What is the measure of the base of the triangle?

length, b_1, and measured height, h_1, the expressions for their areas are the same, hence, their areas must be equal. Therefore, to show that two triangles have the same area, it is sufficient to show that their base lengths and their heights are equal.



Consider as the base the side of the triangle to which the median is drawn. Then, the base of the triangle is divided into two equal segments by the median. The bases of the two triangular regions created by the median are equal; i.e., $\overline{AM} = \overline{MC}$. The median does not affect the height $\overline{BN}$; both regions have the same height to the bases $\overline{AM}$ and $\overline{MC}$, respectively. Since the two triangular regions have the same height and base length, they have the same area.

Given: $\overline{BM}$ is a median of $\triangle ABC$.

Prove: Area of $\triangle ABM$ = Area of $\triangle CBM$.

Statements	Reasons
1. $\overline{BM}$ is the median of $\triangle ABC$.	1. Given.
2. M is the midpoint of $\overline{AC}$.	2. Definition of the median of a triangle.
3. AM = MC.	3. Definition of the midpoint of a line segment.
4. Locate N in $\overleftrightarrow{AC}$ so that $\overline{BN} \perp \overleftrightarrow{AC}$.	4. Through a point external to a line, there is one and only one line perpendicular to the given line.
5. $\overline{BN}$ is an altitude of both $\triangle ABM$ and $\triangle CBM$.	5. Definition of an altitude of a triangle.
6. Area of $\triangle ABM$ = Area of $\triangle CBM$.	6. Two triangles have equal areas if their bases have the same length and the altitudes to their bases have the same length.

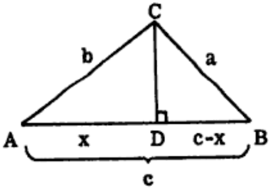
• PROBLEM 8-8

The semiperimeter s of a triangle of sides a,b, and c is defined as half the perimeter, or,

$$\frac{a+b+c}{2} .$$

Find an expression for the area of $\triangle ABC$ and show that it is equivalent to $\sqrt{s(s-a)(s-b)(s-c)}$.

Solution: The area of a triangle equals $\frac{1}{2}Q \cdot H$ where Q is the length of the base and H is the altitude. If, in $\triangle ABC$, we choose $\overline{AB}$ as



the base, then $Q = AB = c$. Then the altitude H must be $\overline{CD}$, or

$$H = \overline{CD} = \frac{1}{2c}\sqrt{(b+a+c)(b+c-a)(a+c-b)(a+b-c)} \quad .$$

The area, therefore, equals $\frac{1}{2}Q \cdot H$ or

$$\frac{1}{2}(c)\left[\frac{1}{2c}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}\right]$$

or

$$\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \quad .$$

Consider the expression $\sqrt{s(s-a)(s-b)(s-c)}$:

if $s = \frac{a+b+c}{2}$; $s-a = \frac{a+b+c}{2} - a = \frac{b+c-a}{2}$;

$s-b = \frac{a+b+c}{2} - b = \frac{a+c-b}{2}$; and $s-c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$.

Then, $$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)}$$

Factoring out the $\frac{1}{2}$'s, we have

$$\sqrt{\frac{1}{16}(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

or

$$\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \quad .$$

Thus, the two expressions we have found are equivalent. Therefore, Heron's Formula, $\sqrt{s(s-a)(s-b)(s-c)}$ is a valid formula for the area of a triangle, if

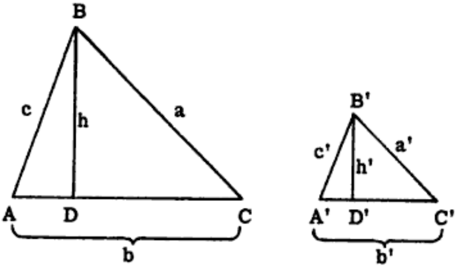
$$s = \frac{a+b+c}{2}$$

where a,b, and c are the lengths of the sides of the triangle.

• PROBLEM 8-9

Show that the ratio of the areas of two similar triangles equals the square of their ratio of similitude.

Solution: Suppose each side of $\Delta 1$ is a times larger than the corresponding sides of $\Delta 2$. Then, Area of $\Delta 2 = \frac{1}{2}b_2h_2$ and the area of $\Delta 1 = \frac{1}{2}b_1h_1$. Since $b_1 = ab_2$ and $h_1 = ah_2$, area of $\Delta 1 = \frac{1}{2}(ab_2)(ah_2) = a^2(\frac{1}{2}b_2h_2)$ or a^2 times larger than the area of $\Delta 2$.



Given: $\Delta ABC \sim \Delta A'B'C'$; $\overline{BD}$ and $\overline{B'D'}$ are the altitudes of their respective triangles. The lengths are indicated in the figure.

Prove: $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta A'B'C'} = \left(\frac{b}{b'}\right)^2$ where $\frac{b}{b'}$ is the ratio of similitude.

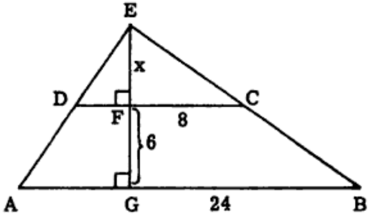
Statements	Reasons
1. $\Delta ABC \sim \Delta A'B'C'$; $\overline{BD}$ and $\overline{B'D'}$ are the altitudes of their respective triangles. $\measuredangle BDC$ and $\measuredangle B'D'C'$ are right angles.	1. Given.
2. $\frac{a}{a'} = \frac{b}{b'}$	2. The sides of similar triangles are proportional.
3. Area of $\Delta ABC = \frac{1}{2}bh$ Area of $\Delta A'B'C' = \frac{1}{2}b'h'$	3. The area of a triangle of base length b and altitude h equals $\frac{1}{2}bh$.
4. $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta A'B'C'} = \frac{\frac{1}{2}bh}{\frac{1}{2}b'h'} = \frac{bh}{b'h'}$	4. Multiplicative Property.
5. $\measuredangle BDC \cong \measuredangle B'D'C'$.	5. All right angles are congruent.
6. $\measuredangle C \cong \measuredangle C'$.	6. Corresponding angles of similar triangles are congruent.
7. $\Delta BDC \sim \Delta B'D'C'$.	7. If two pairs of corresponding angles of two triangles are congruent, then the triangles are similar.
8. $\frac{a}{a'} = \frac{h}{h'}$.	8. The sides of similar triangles are proportional.
9. $\frac{h}{h'} = \frac{b}{b'}$.	9. Transitive Property (Step 2).
10. $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta A'B'C'} = \frac{b^2}{(b')^2} = \left(\frac{b}{b'}\right)^2$	10. Substitution Postulate.

• PROBLEM 8-10

In trapezoid ABCD, as in the accompanying diagram, the larger base, $\overline{AB}$, measures 24 in., the smaller base, $\overline{DC}$, measures 8 in., and the altitude, $\overline{FG}$, measures 6 in.. The nonparallel sides $\overline{AD}$ and $\overline{BC}$ are extended to meet at E.

a) In triangle DEC: (a) Find EF, the measure of the altitude from E to $\overline{DC}$;

b) Find the area of ΔDEC.



Solution: In the trapezoid ABCD, $\overline{DC} \parallel \overline{AB}$. Then, $\Delta DEC \sim \Delta AEB$, because a line parallel to one side of a triangle, which intersects the other two sides at different points, creates a triangle similar to the given triangle. Therefore, by corresponding parts, the proportion

$$\frac{\text{length of altitude } \overline{EF}}{\text{length of altitude } \overline{EG}} = \frac{\text{length of base } \overline{DC}}{\text{length of base } \overline{AB}}$$

is valid. Let x = EF, then x + 6" = EG. By substitution,

$$\frac{x}{x+6''} = \frac{8''}{24''} = \frac{8}{24} .$$

Since, in a proportion, the product of the means equals the product of the extremes,

$$\begin{aligned} 24x &= 8x + 48'' \\ 16x &= 48'' \\ x &= 3'' . \end{aligned}$$

Therefore, the altitude of ΔDEC to $\overline{DC}$, $\overline{EF}$, measures 3 in.

b) The area, A, of $\Delta DEC = \frac{1}{2}bh = \frac{1}{2}DC \times EF$. By substitution,

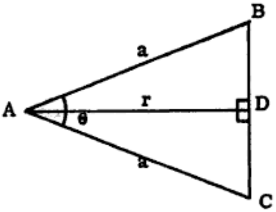
$$A = \tfrac{1}{2}(8 \text{ in.} \times 3 \text{ in.})^2 = 12 \text{ in.}^2 .$$

Therefore, the area of ΔDEC = 12 sq. in.

• PROBLEM 8-11

Let the two congruent sides of an isosceles triangle have lengths a, and let the included angle have measure θ. Choosing the third side as a base, let the corresponding altitude be of length r. Prove that the area enclosed by the triangle is given by the formula

$$A = r^2 \tan \theta/2 .$$



Solution: The figure shows isosceles ΔABC, with altitude $\overline{AD}$ of length r. By definition of an altitude, $\overline{AD} \perp \overline{BC}$, as indicated. We will use the general formula for the area of a triangle, and "tailor" it to the special properties of an isosceles triangle to prove that $A = r^2 \tan \theta/2$.

The area, A, of any triangle with base length b and corresponding altitude length h is

$$A = \tfrac{1}{2}bh.$$

In the ΔABC, the length of the base is BC, and the length of the altitude $\overline{AD}$ is r. Hence,

$$A = \tfrac{1}{2}(BC)r\ .$$

It is now necessary to find BC in terms of the known quantities r and θ. In order to do this, we must digress for a moment.

We note that

$$BC = BD + DC \tag{1}$$

Therefore, if we can find BD and DC, in terms of r and θ, our problem will be solved. We will do this by proving $\Delta ADB \cong \Delta ADC$, and concluding that $BD = DC$. Finding either BD or DC in terms of r and θ, and using (1), we will have finished the problem.

First, note that ΔABC is an isosceles triangle. By definition, this means that $AB = AC$. Furthermore, $AD = AD$. Now, the sum of the measures of the angles of any triangle is 180°. Applying this fact to ΔABD and ΔACD,

$$\begin{aligned} m\measuredangle BAD + m\measuredangle ADB + m\measuredangle DBA &= 180^\circ \\ m\measuredangle CAD + m\measuredangle ADC + m\measuredangle DCA &= 180^\circ\ . \end{aligned} \tag{2}$$

But $m\measuredangle ADB = m\measuredangle ADC = 90^\circ$, and $m\measuredangle DBA = m\measuredangle DCA$. (The last equality follows from the fact that ΔABC is an isosceles triangle.) Using these facts, (2) may be written

$$\begin{aligned} m\measuredangle BAD + 90^\circ + m\measuredangle DBA &= 180^\circ \\ m\measuredangle CAD + 90^\circ + m\measuredangle DBA &= 180^\circ\ . \end{aligned} \tag{3}$$

Comparing the 2 equations in (3), we conclude that $m\measuredangle BAD = m\measuredangle CAD$, or $\measuredangle BAD \cong \measuredangle CAD$. By the S.A.S. (side-angle-side) Postulate, $\Delta ADB \cong \Delta ADC$. Hence, $\overline{BD} \cong \overline{DC}$ and $BD = DC$, upon which (1) becomes

$$BC = 2BD\ . \tag{4}$$

Using (4) in the formula $A = \tfrac{1}{2}(BC)r$ yields

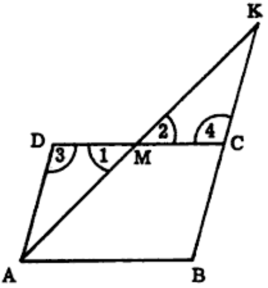
$$A = (BD)r. \tag{5}$$

We may write BD in terms of θ and r by introducing the concept of the tangent of an angle (abbreviated tan). The tangent of an angle of a right triangle is defined as the ratio of the length of the side of the triangle opposite the angle and the length of the side of the triangle adjacent to the angle, so long as neither of the sides are the hypotenuse. For instance, in the figure,

• PROBLEM 8-13

In parallelogram ABCD, M is the midpoint of side $\overline{DC}$, as shown in the figure. Line segment $\overline{AM}$, when extended, intersects $\overleftrightarrow{BC}$ at K.

a) Prove that triangle ADM is congruent to triangle KCM.
b) Prove that triangle AKB is equal in area to parallelogram ABCD.



Solution: a) In this proof, we will prove congruence of the two triangles by the A.S.A. ≅ A.S.A. method.

Statements	Reasons
1. ABCD is a ▱	1. Given.
2. $\overleftrightarrow{KA}$ and $\overleftrightarrow{KB}$ are straight lines	2. Given.
3. ∢ 1 ≅ ∢ 2	3. If two angles are vertical angles then they are congruent.
4. $\overline{AD} \parallel \overline{BC}$.	4. The opposite sides of a parallelogram are parallel.
5. ∢ 3 ≅ ∢ 4	5. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.
6. $\overline{DM} \cong \overline{CM}$	6. A midpoint divides a line into two congruent segments.
7. ΔADM ≅ ΔKCM.	7. A.S.A. ≅ A.S.A.

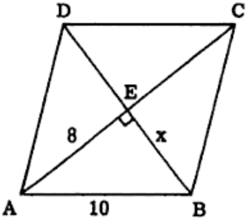
b) To prove that the area of ΔAKB is equal to the area of ▱ABCD, we must show that the sum of the areas of quadrilateral ABCM and ΔKCM is equal to the sum of the areas of quadrilateral ABCM and ΔADM. The first sum is precisely the area of ΔAKB; the second sum is the area of ▱ ABCD.

Statements	Reasons
1. ΔKCM ≅ ΔADM.	1. Proved in part (a).
2. Area of ΔKCM = Area of ΔADM.	2. If two triangles are congruent, then they are equal in area.
3. Area of ABCM = Area of ABCM.	3. Reflexive Property of Equality.
4. Area of ΔKCM + Area of ABCM = Area of ΔADM + Area of ABCM.	4. Equal quantities added to equal quantities sum to equal quantities (see step (2)).
5. Area of ΔKCM + Area of ABCM = Area of ΔAKB and Area of ΔADM + Area of ABCM = Area of ▱ABCD.	5. If a polygon which encloses a region is separated into several polygons which do not overlap,

	then its area is the sum of the areas of these several polygons. (Area-Addition Postulate).
6. Area of $\triangle AKB$ = Area of ▱ABCD.	6. Substitution Postulate.

• PROBLEM 8-14

Find the area of a rhombus, each of whose sides is 10 in., and one of whose diagonals is 16 in.



Solution: Since the area of a rhombus is equal to one-half the product of the lengths of the diagonals, we must determine the length of the second diagonal before the area can be calculated. (It has been represented in the accompanying figure).

Since the diagonals of a rhombus bisect each other at right angles, we can conclude that $\overline{AE} \perp \overline{BD}$, and that $AE = EC = \frac{1}{2}AC$. $\overline{AC}$ is assumed to be the given 16 in. diagonal. Therefore,

$$AE = \tfrac{1}{2}(16 \text{ in.}) = 8 \text{ in.}$$

Because $\triangle AEB$ is a right triangle we can apply the Pythagorean Theorem to determine the length of half-diagonal $\overline{EB}$. Let $x = EB$. Then,

$$x^2 + (8 \text{ in.})^2 = (10 \text{ in.})^2$$

$$x^2 + 64 \text{ in.}^2 = 100 \text{ in.}^2$$

$$x^2 = 36 \text{ in.}^2$$

$$x = 6 \text{ in.}$$

Therefore, the whole diagonal $\overline{BD}$ measures twice EB or $BD = 2EB = 2(6 \text{ in.}) = 12$ in.

Area of a rhombus $= \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the rhombus diagonals. By substitution,

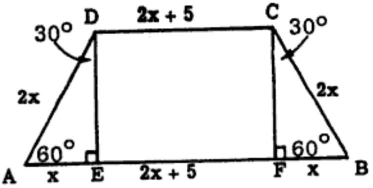
$$\text{Area} = \tfrac{1}{2}(12)(16)\text{in.}^2$$

$$= 6(16)\text{in.}^2 = 96 \text{ in.}^2$$

Therefore, the area of rhombus ABCD = 96 sq. in.

• PROBLEM 8-15

In the accompanying figure, $\overline{AB}$ and $\overline{DC}$ are the bases of trapezoid ABCD. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect in E. Prove that triangle ADE is equal in area to triangle BCE.



Solution: To find the area, we must first determine the length of both bases and the length of the altitude of trapezoid ABCD.

Triangle AED is a right triangle; and since $m\angle DAE = 60^\circ$, then the $m\angle ADE = 30^\circ$.
In a 30-60-90 triangle, the length of the side opposite the 30° angle equals ½ the length of the hypotenuse.

Therefore, $AD = 2AE$. Let $AE = x$. Then, $AD = 2x$. We are told that $\overline{CD}$ exceeds $\overline{AD}$ by 5. Therefore, $CD = 2x + 5$.

Since DCFE is a rectangle, $EF = CD = 2x + 5$.

Because ABCD is isosceles, $\Delta DAE \cong \Delta CBF$, by A.S.A. ≅ A.S.A.

($AD = BC = 2x$; $m\angle ADE = m\angle BCF = 30^\circ$; and $m\angle DAE = m\angle CBF = 60^\circ$.)

By corresponding parts, $AE = FB = x$.
Referring to the diagram, we see that $\overline{AE} + \overline{EF} + \overline{FB} + \overline{BC} + \overline{CD} + \overline{DA} =$ perimeter of ABCD. By substitution,

$$\begin{aligned} x + (2x+5) + x + 2x + (2x+5) + 2x &= 110 \\ 10x + 10 &= 110 \\ 10x &= 100 \\ x &= 10\ . \end{aligned}$$

Therefore, the length of base $\overline{CD} = 2(10) + 5$, or 25. The length of base $\overline{AB} = 10 + (2(10) + 5) + 10 = 45$.
To calculate the altitude DE of trapezoid ABCD we apply the Pythagorean Theorem to right triangle AED. Let h = length of altitude $\overline{DE}$.

We have found that $AD = 20$ and $AE = 10$. By substitution,

$$\begin{aligned} (20)^2 &= h^2 + (10)^2 \\ 400 &= h^2 + 100 \\ h^2 &= 300 \\ h &= \sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3}\ . \end{aligned}$$

Therefore, since the area of a trapezoid $= \frac{1}{2}h(b+b')$, by plugging in the values we have determined, we obtain

$$\begin{aligned} A &= \tfrac{1}{2}(10\sqrt{3})(25+45) \\ &= \tfrac{1}{2}(10\sqrt{3})(70) \\ &= 35(10\sqrt{3}) \\ &= 350\sqrt{3}\ . \end{aligned}$$

Therefore, the area of the trapezoid is $350\sqrt{3}$.

AREA OF A CIRCLE

• PROBLEM 8-17

The circumference of a tree trunk is 6.6 ft.

a) What is the diameter of the trunk?

b) What is the area of a cross-section of the tree trunk?

Solution: To a first approximation, we may assume that the tree trunk is cylindrical, and that its cross-section is circular.

a) The circumference, C, of a circle of diameter, d, is

$$C = \pi d.$$

In our case, we want d and we know C. Hence,

$$d = \frac{C}{\pi} = \frac{6.6}{3.14}\text{ ft.} = 2.10\text{ ft.}$$

b) The area, A, of a circle of radius, r, is

$$A = \pi r^2 \qquad (1)$$

Since $2r = d$, where d is the circle's diameter, we may rewrite (1) as

$$A = \pi(d/2)^2 = \frac{\pi d^2}{4}$$

Using the value of d found in part (a), we have

$$A = \frac{(3.14)}{(4)}(2.10\text{ ft})^2$$

$$A = 3.46\text{ ft.}^2$$

• PROBLEM 8-18

Find the area of a circle of radius 4. What is the area of the sector subtended by a central angle of 45°.

Solution: The area of the circle is πr^2. Therefore, for a circle of radius 4, the area is $\pi(4)^2 = 16\pi$.

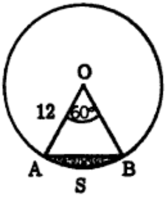
The area of a sector with radius r and a central angle of measure n equals

$$\frac{n}{360}\cdot\pi r^2 .$$

In this case, $n = 45^\circ$ and area equals $\frac{45}{360}\cdot\pi(4)^2 = \frac{1}{8}\cdot\pi 16 = 2\pi$.

• PROBLEM 8-19

In a circle whose radius is 12, find the area of a minor segment whose arc has a central angle of 60°. [Leave the answer in terms of π, and in radical form.]



Solution: A segment of a circle is the union of an arc of the circle, its chord and the region bounded by the arc and the chord. While there is no formula for the area of a segment, a general rule that can be applied is to subtract the area of the triangle formed by the rays of the central angle and chord of the arc from the area of the sector bounded by the rays of the central angle and the arc. In the accompanying figure, Area of segment ASB = Area of sector OASB - area of triangle AOB.

$$\text{Area of sector } OASB = \frac{\text{measure of central angle}}{360^\circ} \times \pi r^2 ,$$

where r is the radius of the circle O. By substitution,

$$\text{Area of sector } OASB = \frac{60^\circ}{360^\circ} \times \pi(12)^2 = \frac{1}{6}(144)\pi = 24\pi .$$

Since $\overline{OA}$ and $\overline{OB}$ are radii of a circle, $\overline{OA} \cong \overline{OB}$, and, since the angle included between them is given to be 60°, $\triangle AOB$ is equilateral. As such, its area equals $\frac{s^2}{4}\sqrt{3}$, where s is the length of 1 side of the $\triangle$. We are told $s = 12$ and, by substitution,

$$\text{Area of } \triangle AOB = \frac{(12)^2}{4}\sqrt{3} = \frac{144}{4}\sqrt{3} = 36\sqrt{3} .$$

Therefore,

$$\text{Area of segment } ASB = \text{Area of sector } OASB - \text{Area of triangle } AOB$$
$$= 24\pi - 36\sqrt{3} .$$

Therefore, Area of segment $ASB = 24\pi - 36\sqrt{3}$.

• PROBLEM 8-20

The ratio of the area of two circles is 16:1. If the diameter of the smaller circle is 3, find the diameter of the larger circle.

Solution: The area of a circle equals πr^2 and, from this, we can conclude the ratio of areas is equal to the square of the ratio of radii length. However, we know that $r = \frac{1}{2}d$, where d = diameter, therefore, by substitution,

$$A = \pi(\tfrac{1}{2}d)^2 = \tfrac{1}{4}\pi d^2 .$$

This shows the ratio of any two areas will be equal to the square of the ratio of the diameters. Let $A/A' =$ the ratio of areas. Also, $d' =$ the smaller diameter, and $d =$ the larger diameter. Then,

$$\frac{A}{A'} = \left(\frac{d}{d'}\right)^2 .$$

By substitution,

$$\frac{16}{1} = \left(\frac{d}{3}\right)^2$$

$$\frac{16}{1} = \frac{d^2}{9}$$

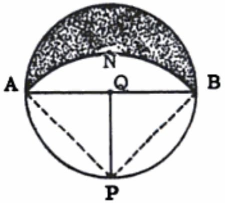
$$144 = d^2 \Longrightarrow d = 12 .$$

Therefore, the diameter of the larger circle is 12.

• PROBLEM 8-21

Given: P is a point on $\widehat{AB}$ of circle Q, so that $\widehat{AP} \cong \widehat{BP}$; $\widehat{ANB}$ is an arc of circle P; $\overline{AQB}$.

Prove: The area of the shaded region $= (PQ)^2$.



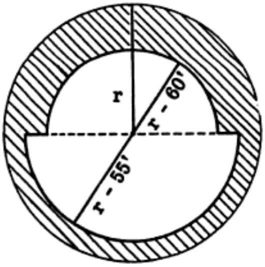
Solution: To find the area of any irregular region, look for areas that can be calculated and that either sum or form, by their difference, the desired region. Here, the shaded region is the difference of semicircle Q minus the region ANBQ. The area of semicircle Q can be found with the formula $\frac{1}{2}\pi r^2$. The region ANBQ is the difference of sector PANB and ΔAPB. Therefore, the area of region ANBQ equals the area of the sector PANB minus the area of ΔAPB.

Statements	Reasons
1. P is a point on $\widehat{AB}$ of circle Q, so that $\widehat{AP} \cong \widehat{BP}$; $\widehat{ANB}$ is an arc of circle P; $\overline{AQB}$.	1. Given.
2. AB is a diameter.	2. Any chord that passes through the center of the circle is a diameter.
3. $m\measuredangle AQB = 180^\circ$.	3. The measure of a straight angle is 180° .
4. $\measuredangle AQP \cong \measuredangle PQB$.	4. Central angles that intercept congruent arcs are congruent.
5. $m\measuredangle AQP + m\measuredangle PQB = m\measuredangle AQB$.	5. Angle Addition Postulate.
6. $2m\measuredangle AQP = 180$ or $m\measuredangle AQP = m\measuredangle PQB = 90^\circ$.	6. Substitution Postulate.
7. $AQ = QB = PQ$.	7. All radii are congruent.

8. $PB^2 = BQ^2 + PQ^2 = 2 \cdot PQ^2$.	8. Pythagorean Theorem and Substitution Postulate.
9. $\triangle APB$ is a right triangle.	9. Any triangle inscribed in a semicircle with the diameter as a side is a right triangle, with the hypotenuse being the diameter.
10. $\sphericalangle APB$ is a right angle.	10. Definition of right triangle.
11. Area of sector PANB = $\frac{90}{360} \cdot \pi \cdot PB^2$.	11. The area of a sector with radius r and central angle n is $\frac{n}{360} \cdot \pi r^2$.
12. Area of sector PANB = $\frac{1}{4}\pi(2PQ^2) = \frac{1}{2}\pi PQ^2$.	12. Substitution Postulate and Step 8.
13. Area of $\Delta APB = \frac{1}{2}AB \cdot QP$.	13. The area of a triangle with base b and altitude h is $\frac{1}{2}bh$.
14. Area of $\Delta APB = \frac{1}{2}PQ \cdot (2PQ) = (PQ)^2$.	14. Substitution Postulate.
15. Area of region ANBQ = (Area of sector PANB) - (Area of $\Delta APB) = \frac{1}{2}\pi PQ^2 - PQ^2$.	15. The area of non-overlapping regions is the sum of the areas of each region. Also, Substitution Postulate.
16. Area of semicircle with diameter $\overline{AQB} = \frac{1}{2}\pi(AQ)^2$.	16. Area of a semicircle of radius r equals $\frac{1}{2}\pi r^2$.
17. Area of shaded area = (Area of semicircle $\overline{AQB}$) - (Area of region ANBQ).	17. The area of non-overlapping regions is the sum of the areas of each region.
18. Area of shaded region = $\frac{1}{2}\pi(AQ)^2 - (\frac{1}{2}\pi PQ^2 - PQ^2)$.	18. Substitution Postulate.
19. $(\frac{1}{2}\pi - \frac{1}{2}\pi + 1)PQ^2 = PQ^2$.	19. Substitution Postulate and Factoring.

• PROBLEM 8-22

A farmer is cutting a field of oats with a machine which takes a 5 ft. cut. The field he is cutting is circular and when he has been round it 11½ times (starting from the perimeter) he calculates that he has cut half the area of the field. How large is the field? (Answer to the nearest 100 sq. yds.)



Solution: In the accompanying figure, the shaded area is the area already cut; r is the radius of the field. Originally the entire field was uncut and the uncut area was a circle of radius r. The first trip around, he reduces the uncut area from a circle of radius r to a circle of radius r - 5. On the nth trip, he reduces the uncut area from a circle of radius r - 5(n-1) to a circle of radius r - 5n. Thus, on the 11th trip, the uncut area is a circle of radius r - 55. Because the farmer made 11 and ½ trips, the uncut area consists of a semicircle of radius r - 55 and a semicircle of radius r - 60.

We are given that the area of the uncut portion equals the area of the cut portion. Thus, the area of the unplowed area is one-half the area of the field. The area of the uncut portion is the sum of the areas of the semicircle of radius r - 55 and the area of semicircle of radius r - 60. Given that the formula of the semicircle of radius x is $\frac{1}{2}\pi x^2$, we have

(i) area of semicircle of r - 55 = $\frac{1}{2}\pi(r-55)^2$

(ii) area of semicircle of r - 60 = $\frac{1}{2}\pi(r-60)^2$

(iii) area of unplowed area = $\frac{1}{2}\pi(r-55)^2 + \frac{1}{2}\pi(r-60)^2$

$$= \frac{1}{2}\pi((r-55)^2 + (r-60)^2)$$

(vi) area of field = 2 · area of unplowed area.

(v) $\pi r^2 = 2 \cdot \frac{1}{2}\pi((r-55)^2 + (r-60)^2)$

(vi) $r^2 = r^2 - 110r + 3025 + r^2 - 120r + 3600$

(vii) $r^2 - 230r + 6625 = 0$

Using the quadratic formula,

$$ax^2 + bx + c = 0, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we obtain for the radius of the field.

(viii) $$r = \frac{230 \pm \sqrt{230^2 - 4(6625)}}{2} = \frac{230 \pm \sqrt{26400}}{2}$$

(ix) $$r = \frac{230 \pm 162.48}{2} = 196.24 \text{ or } 33.76.$$

Note that the answer r = 33.76 makes no sense since this would imply that the radius of the unplowed portion would be r = 33.76 - 55 = -21.24. Thus, r = 196.24 $\approx$ 196 ft. is the only solution.

To find the area of the circular field, remember that the area of the circle is πr^2.

(x) Area of field = $\pi(196)^2 = 38416\pi$

$38416\pi \approx 120687$ sq. ft.

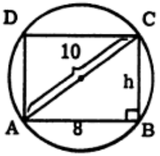
Since 1 sq. ft. = $\frac{1}{9}$ sq. yd.,

(xi) Area of field = $120687(\frac{1}{9}$ sq. yd.) = 13,400 sq. yd.

AREA BY THE PYTHAGOREAN THEOREM

• PROBLEM 8-23

A rectangle is inscribed in a circle whose radius is 5 inches. The base of the rectangle is 8 inches. Find the area of the rectangle. (See figure.)



Solution: The diagonal of the rectangle is a diameter of the circle and as such, each of the two triangles shown in the figure is inscribed in a semi-circle. Therefore, they are right triangles.

Since ΔBAC is a right triangle, we can use the Pythagorean Theorem to determine the altitude, h, of ABCD, as indicated in the figure. We will need this to calculate the area of the rectangle. Therefore,

$$\begin{aligned} h^2 + (8 \text{ in.})^2 &= (10 \text{ in.})^2 \\ h^2 + 64 \text{ in.}^2 &= 100 \text{ in.}^2 \\ h^2 &= 36 \text{ in.}^2 \quad \text{and} \\ h &= 6 \text{ in.} \end{aligned}$$

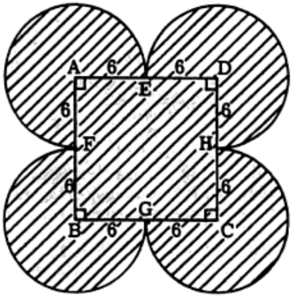
Now, Area = bh. Let b = 8 in., h = 6 in. By substitution,

$$A = (8 \times 6) \text{in.}^2 = 48 \text{ in.}^2$$

Therefore, the area of the inscribed rectangle is 48 sq. in.

• PROBLEM 8-24

Find, in terms of π, the area of the shaded region in the diagram.



Solution: The shaded region consists of square ABCD, whose side is 12, and four major sectors whose areas are equal because their radii and central angles are equal. We will calculate the area of the square and the major sectors, and then sum them, to determine the area of the shaded region.

$$\text{Area of the square} = (AB)^2 = (12)^2 = 144.$$

Each sector spans 270°, since the corners of the square cut off arc lengths that measure 90° from each circle. Since the area of a sector is proportional to the central angle, the area of each sector is angle bisector of each vertex angle. Because E is on the bisector of ∢ DAB, E is equidistant from sides $\overline{AD}$ and $\overline{AB}$. Because E is on the bisector of ∢ ABC, E is equidistant from sides $\overline{AB}$ and $\overline{BC}$. Because E is on the bisector of ∢ BCD, E is equidistant from sides $\overline{BC}$ and $\overline{DC}$. Combining these three facts and using the transitive property, we have that E is equidistant from all four sides. Therefore, E, the center of the inscribed circle, is the same point as E , the intersection of the diagonals.

To show the second step, that ΔDEC is a right triangle, we remember that the diagonals of a rhombus are perpendicular to each other. Therefore, ∢ DEC is a right angle and ΔDEC is a right triangle.

In right ΔDEC, we have the length of the hypotenuse, DC = 25. Because the diagonals of a parallelogram bisect each other, we have that the measure of the leg DE = $\frac{1}{2}$DB = $\frac{1}{2}(40) = 20$. By the Pythagorean Theorem,

$$EC^2 = DC^2 - DE^2 = 25^2 - 20^2 = 225;$$

or

$$EC = 15.$$

We wish to find the radius EF. EF, as the altitude to the hypotenuse, is the mean proportional of the hypotenuse segments, DF and FC. To find DF and FC, remember that the adjacent leg is the mean proportional of the segment and the hypotenuse. Thus,

$$\frac{DF}{DE} = \frac{DE}{DC} \quad \text{or} \quad DF = \frac{DE^2}{DC} = \frac{20^2}{25} = 16.$$

Then, DF = 16 and $\overline{FC}$ = DC - DF = 25 - 16 = 9.

Since altitude $\overline{EF}$ is the mean proportional of the base segments, we obtain

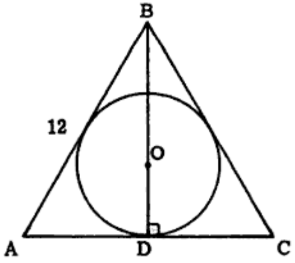
$$\frac{DF}{EF} = \frac{EF}{FC} \quad \text{or} \quad EF^2 = DF \cdot FC = 16 \cdot 9 = 144.$$

Then $EF = \sqrt{144} = 12$.

Since the radius of the inscribed circle, EF, equals 12, the area of the circle = $\pi r^2 = \pi(12)^2 = 144\pi$.

• PROBLEM 8-26

A circle is inscribed in an equilateral triangle, whose side is 12. Find, to the nearest integer, the difference between the area of the triangle and the area of the circle. (Use $\pi = 3.14$ and $\sqrt{3} = 1.73$.)



Solution: The diameter of the inscribed circle equals the side of the square. Therefore, the area equals $\pi r^2 = \pi(4)^2 = 16\pi$.

The diameter of the circumscribed circle is the diagonal of the square. The diagonal of a square is the hypotenuse of the right triangle formed by any two sides of the square adjacent at a vertex not intersected by the diagonal. Its length can be found by applying the Pythagorean Theorem, since we know the length of two sides of right ΔABC.

$$(AC)^2 + (AB)^2 = (BC)^2$$

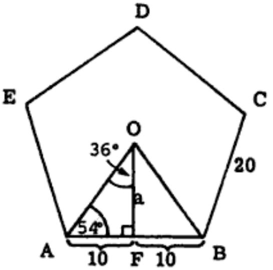
$AC = AB = 8$ and accordingly, by substitution,

$$\begin{aligned} (8)^2 + (8)^2 &= (BC)^2 \\ 2(64) &= (BC)^2 \\ BC &= \sqrt{128} = \sqrt{64}\,\sqrt{2} = 8\sqrt{2} \end{aligned}$$

The diagonal equals $8\sqrt{2}$, therefore, $r = 4\sqrt{2}$. The area of the circle equals $\pi(4\sqrt{2})^2 = 32\pi$. The area of the circumscribed circle exceeds the area of the inscribed circle by $32\pi - 16\pi = 16\pi$. In general, for a given square, the area of the circumscribing circle is twice the area of the inscribed circle.

• PROBLEM 8-28

The side of a regular pentagon is 20 inches in length. (a) Find, to the nearest tenth of an inch, the length of the apothem of the pentagon. (b) Using the result obtained in part (a), find, to the nearest ten square inches, the area of the pentagon.



Solution: (a) When the central angle AOB is drawn, as shown in the figure, its rays, along with side $\overline{AB}$ of pentagon ABCDE form a triangle. The central-angle theorem states that the measure of a central angle of a regular polygon equals 360° divided by the number of sides of the polygon. Therefore, the measure of $\measuredangle AOB$ is 360°/5, or 72°, where 5 is the number of sides in the pentagon.

By theorem, apothem $\overline{OF}$ bisects central angle AOF and pentagon side $\overline{AB}$. Since $\overline{OF} \perp \overline{AB}$, therefore, ΔOFA is a right triangle.

$$m\measuredangle AOF = \tfrac{1}{2}(m\measuredangle AOB) = \tfrac{1}{2}(72^\circ) = 36^\circ \ .$$

Therefore, $m\measuredangle FAO = 180^\circ - (m\measuredangle AOF + m\measuredangle OFA)$.

By substitution, $m\measuredangle FAO = 180^\circ - (36^\circ + 90^\circ) = 54^\circ$. We also know,

$\overline{AF} = \frac{1}{2}\overline{AB} = \frac{1}{2}(20 \text{ in.}) = 10$ in.

Since ΔOAF is a right triangle, knowing an angle measure and the length of its adjacent side, we can calculate the length of $\overline{OF}$ by using the tangent ratio.

$$\tan \measuredangle OAF = \frac{\text{length of leg opposite } \measuredangle OAF}{\text{length of leg adjacent } \measuredangle OAF} .$$

Let a = length of leg opposite $\measuredangle OAF$ (see figure.) Then, by substitution,

$$\tan 54^\circ = \frac{a}{10 \text{ in.}} .$$

From a standard tangent table, we find $\tan 54^\circ = 1.3764$. Therefore,

$$1.3764 = \frac{a}{10 \text{ in.}}$$

$$a = (10 \text{ in.})(1.3764) = 13.764 \text{ in.}$$

The length of apothem $\overline{OF}$ is 13.8 in.

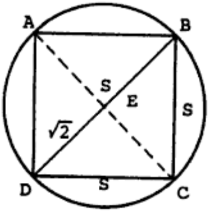
(b) The area of a regular polygon is given by $A = \frac{1}{2}$(apothem length) (perimeter) $= \frac{1}{2}ap$. The perimeter, P, of pentagon $ABCDE = 5(20)$in. = 100 in. Therefore, by substitution,

$$A = \tfrac{1}{2}(13.8)(100)\text{in.}^2 = (50)(13.8)\text{in.}^2 = 690 \text{ in.}^2 .$$

Therefore, the area of pentagon ABCDE is 690 sq. in.

• PROBLEM 8-29

Draw a circle circumscribed about a square of edge length s. What is the area of the region outside the square but inside the circle?



Solution: Let the area of the region inside the circle but outside the square be A(R). Furthermore, let the area of the circle be A(C) and the area of the square A(s). From the figure, we have

(1) $A(R) = A(C) - A(s)$.

Since the square is of edge length s, $A(s) = s^2$. To find $A(C) = \pi r^2$, we must first find the radius r.

We first show that the diameter of the circle equals the diagonal of the square. Let E be the center of the circumscribing circle. To show $\overline{DB}$ is a diameter, we must show $\overline{DEB}$ is a straight line. We will show this by showing that $\measuredangle DEA$ and $\measuredangle BEA$ form a straight angle. From an earlier theorem, we know that the central angles of a regular polygon are congruent. Therefore, $m\measuredangle DEA = m\measuredangle AEB = m\measuredangle BEC = m\measuredangle CED = \frac{360^\circ}{4} = 90^\circ$. $m\measuredangle DEA + m\measuredangle BEA = 90^\circ + 90^\circ = 180^\circ$. Since $m\measuredangle DEA + m\measuredangle BEA = m\measuredangle DEB$, $m\measuredangle DEB = 180^\circ$, then $\measuredangle DEB$ is a straight angle and D, E, and B are collinear. Thus $\overline{DB}$, the diagonal, is also a diameter.

To find DB, note that ΔDBC is a right triangle.

Using the Pythagorean Theorem, we have $DB^2 = DC^2 + CB^2 = s^2 + s^2 = 2s^2$. Thus, $DB = \sqrt{2s^2} = \sqrt{2}s$.

The radius of the circle equals ½ the diameter DB or $\frac{1}{2}(\sqrt{2}s)$. The area of the circle equals

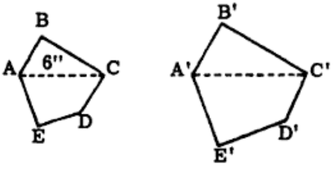
$$\pi r^2 = \pi \left(\frac{\sqrt{2}}{2}\, s\right)^2 = \frac{\pi s^2}{2}\ .$$

Returning to equation (1), we obtain

$$A(R) = \frac{\pi s^2}{2} - s^2 = s^2 \left(\frac{\pi}{2} - 1\right)$$

• PROBLEM 8-30

The area of pentagon ABCDE is 18 sq. in.; the area of a similar pentagon A'B'C'D'E' is 32 sq. in. The diagonal AC is 6 in.; find the length of A'C'.



Solution: For two polygons to be similar, all corresponding angles are congruent, and all corresponding sides must be proportional. We will first show that the ratio of similitude of the diagonals, A'C'/AC, is the same as the ratio of similitude of the sides, A'B'/AB. Then, we find this ratio of similitude of the two polygons using the fact that the ratio of the area of similar polygons equals the square of the ratio of similitude.

To show that A'C'/AC = ratio of similitude, we show

$$\Delta ABC \sim \Delta A'B'C'\ .$$

Note $\measuredangle B \cong \measuredangle B'$ and $\frac{A'B'}{AB} = \frac{B'C'}{BC}$ because ABCDE and A'B'C'D'E' are similar polygons, By the S.A.S. Similarity Theorem, $\Delta ABC \sim \Delta A'B'C'$, and thus $\frac{A'C'}{AC} = \frac{A'B'}{AB}$.

Since $\frac{A'B'}{AB}$ is the ratio of similitude, $\frac{A'C'}{AC}$ = ratio of similitude.

To find the ratio of similitude, remember that

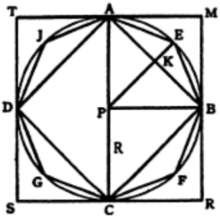
$$(\text{ratio of similitude})^2 = \frac{\text{area of } A'B'C'D'E'}{\text{area of } ABCDE} = \frac{32}{18} = \frac{16}{9}$$

$$\text{ratio of similitude} = \sqrt{\frac{16}{9}} = \frac{4}{3}\ .$$

Thus, $\frac{A'C'}{AC} = \frac{4}{3}$. Since $AC = 6$, then $\frac{A'C'}{6} = \frac{4}{3}$ or $A'C' = \frac{4}{3}(6) = 8$ in.

• PROBLEM 8-31

Prove that the area of an inscribed regular octagon is equivalent to that of a rectangle whose dimensions are the sides of the inscribed and circumscribed squares.



Solution: In the accompanying figure, AEBFCGDJ is the inscribed octagon; ABCD, the inscribed square; MRST, the circumscribed square. We are given circle P with radius of length r.

We will proceed by first determining the area of the octagon. Second, we determine the sides of the inscribed and circumscribed squares, which are the width and length of the rectangle. From this, we calculate the area of the rectangle. If the two areas are equal our work is done.

The area of octagon AEBFCGDJ = area of ABCD + area of $\triangle AEB$ + area of $\triangle BFC$ + area of $\triangle DGC$ + area of $\triangle AJD$.

The area of $ABCD = AB^2$. To find AB, note that it is the hypotenuse of isosceles right triangle APB. The legs are formed by radii of ⊙P. Hence, by the Pythagorean Theorem,

$$AB^2 = r^2 + r^2 = 2r^2 \quad \text{and} \quad AB = r\sqrt{2}\ .$$

The area of $ABCD = (r\sqrt{2})^2 = 2r^2$. By the S.S.S. Postulate, all the triangles are congruent and the expression reduces to area of ABCD + 4(area of $\triangle AEB$), area of $\triangle AEB = \frac{1}{2}$(base)(altitude) $= \frac{1}{2}(AB)(EK)$. We know $AB = r\sqrt{2}$.

$EK = r - KP$. KP can be found and, hence, EK will be known (in terms of r).

Since the octagon is regular, $AE = EB$. In the same circle congruent chords intercept congruent arcs which have congruent central angles. Hence, $\measuredangle APK \cong \measuredangle BPK$ or $\overline{PK}$ is the angle bisector of the vertex angle of isosceles triangle APB. As such, $\overline{PK}$ is the perpendicular bisector of $\overline{AB}$. Therefore, $AK = KB = \frac{1}{2}(AB) = \frac{1}{2}r\sqrt{2}$. ΔAKP is a right triangle. Hence, we can apply the Pythagorean Theorem to find KP, given that we know $AP = r$ and $AK = \frac{1}{2}r\sqrt{2}$,

$$(AP)^2 = (AK)^2 + (KP)^2$$

$$r^2 = (\tfrac{1}{2}r\sqrt{2})^2 + (KP)^2$$

$$r^2 - \tfrac{1}{4}2r^2 = (KP)^2$$

$$\tfrac{1}{2}r^2 = (KP)^2$$

$$\frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}r = KP$$

$$\tfrac{1}{2}r\sqrt{2} = KP$$

As such, $EK = r - \frac{1}{2}r\sqrt{2}$.

$$\begin{aligned}\text{Area of } \Delta AEB &= \tfrac{1}{2}(r\sqrt{2})(r - \tfrac{1}{2}r\sqrt{2})\\ &= \tfrac{1}{2}(r^2\sqrt{2} - \tfrac{1}{2}2r^2) = \tfrac{1}{2}r^2(\sqrt{2} - 1) .\end{aligned}$$

Recall, area of AEBFCGDJ = area of ABCD + 4(area of ΔAEB).

By substitution, $= 2r^2 + 4(\frac{1}{2}r^2(\sqrt{2} - 1))$

$= 2r^2 + 2r^2(\sqrt{2} - 1)$

Therefore, Area of octagon $= 2r^2\sqrt{2}$. We have one side of the required rectangle, $AB = r\sqrt{2}$. We will now find MR, the other side.

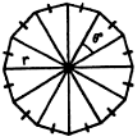
Since MRST is a square, $\overline{MT} \parallel \overline{SR}$ and MR is the perpendicular distance between $\overline{MT}$ and $\overline{SR}$. A and C are points of tangency. As such, $\overline{CA} \perp \overline{MT}$ and $\overline{CA} \perp \overline{SR}$ and CA is also a $\perp$ distance between $\overline{MT}$ and $\overline{SR}$. Since parallel lines are everywhere equidistant, MR = CA. CA is a diameter of OP. Hence, CA = 2r and MR = 2r.

The proposed rectangle has sides 2r and $r\sqrt{2}$. Its area then equals $2r^2\sqrt{2}$. Since this is the area of the octagon, we have shown the desired results.

AREA BY THE TRIGONOMETRIC FUNCTIONS

• PROBLEM 8-32

Using the fact that the area of an n sided regular polygon is $nr^2 \tan \pi/n$, find the area of a circle. (Here, the polygon is composed of n isosceles triangles, with height r. See figure.)



Solution: The figure shows the situation for n = 12. The idea here is that a circle is the limit of a regular polygon as the number of its sides increases without limit. We obtain the desired result by letting n get very large in the formula for the area of an n-sided regular polygon.

First, note that as n gets larger, tangent of π/n gets very small. Now, the tangent of any small angle is approximately equal to the angle itself. Hence,

$$\text{as } n \to \infty, \ \tan \pi/n \to \pi/n.$$

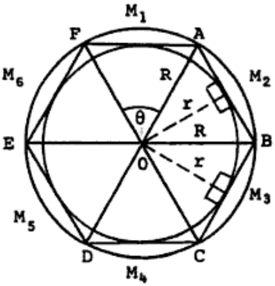
Then, using this in the formula given in the problem

$$\text{as } n \to \infty, \ nr^2 \tan \pi/n \to nr^2 \ \pi/n = \pi r^2 .$$

Hence, the area of a circle of radius r is πr^2 .

• PROBLEM 8-33

Prove that the area bounded by a regular polygon of n sides circumscribed about a circle with a radius of length r is given by the formula $A = nr^2 \tan \pi/n$.



Solution: The figure shows the situation for the case where $n = 6$. In general, we see that we may consider the polygon of n sides to be composed of n triangles (for example, $\triangle AOB$, $\triangle BOC$, etc.). Since the polygon is circumscribed about circle O, $\overline{OM_2} \perp \overline{AB}$. This makes $\overline{OM_2}$, of length r, an altitude of $\triangle AOB$. If $\triangle AOB$ were isosceles, we could find its area by using the formula $A = r^2 \tan \theta/2$. We now prove that $\triangle AOB$ is, indeed, isosceles.

In the figure, we have circumscribed a circle about the polygon, By definition, the center of a regular polygon is also the center of the inscribed and circumscribed circles. Now, look at $\triangle AOB$. $\overline{AO} \cong \overline{BO}$, since both are radii of the larger circle. Hence, $\triangle AOB$ is isosceles. Its area is, then,

$$A = r^2 \tan \theta/2$$

where r and θ are as shown.

Now we prove that each triangle (of which there are n) is congruent to every other triangle of the polygon. Consider $\triangle AOB$ and $\triangle BOC$, $\overline{AO} \cong \overline{CO}$, since both are radii of the large circle. $\overline{OB} \cong \overline{OB}$. $\overline{AB} \cong \overline{CB}$, because the sides of a regular polygon are all congruent. Therefore, by the S.S.S. (side-side-side) Postulate, $\triangle AOB \cong \triangle BOC$. This may be done n times to prove that all the triangles comprising the polygon are congruent. Hence, they all have the same area. Since the n-sided polygon is composed of the n adjacent triangles (see figure), the area of the polygon is the sum of the areas of each of the n triangles. Hence,

$$A_p = nr^2 \tan \theta/2 .$$

But

$$n\theta = 2\pi$$

or

$$\theta/2 = \pi/n .$$

Hence,

$$A_p = nr^2 \tan \pi/n .$$

• PROBLEM 8-34

Find the area of a regular hexagon circumscribed about a circle of radius r. Calculate the area explicitly if
a) $r = 4$; b) $r = 9$; c) $r = 16$; d) $r = 25$.

Solution: The area of a regular n-sided polygon circumscribed about a circle of radius r is given by

$$A = nr^2 \tan \pi/n .$$

In a hexagon, $n = 6$. $\tan \pi/6 = \sqrt{3}/3$

Thus

$$A = 6r^2 \tan \pi/6 = 2r^2 \sqrt{3}$$

a) For $r = 4$, $A = 2(16)\sqrt{3} = 32\sqrt{3}$

b) For $r = 9$, $A = 2(81)\sqrt{3} = 162\sqrt{3}$

c) For $r = 16$, $A = 2(256)\sqrt{3} = 512\sqrt{3}$

d) For $r = 25$, $A = 2(625)\sqrt{3} = 1250\sqrt{3}$.

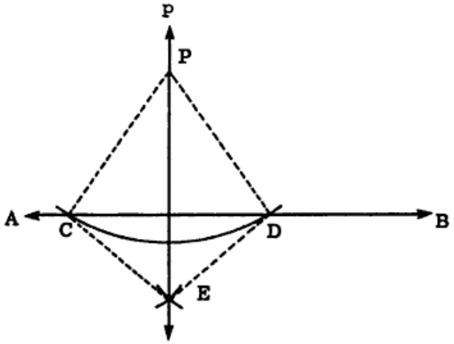
CHAPTER 9

CONSTRUCTIONS

LINES AND ANGLES

• PROBLEM 9-1

Construct a line perpendicular to a given line through a given point outside the line.



Solution: We are given line $\overleftrightarrow{AB}$ and point P, not on $\overleftrightarrow{AB}$, as in the figure shown. We wish to construct a line through P, perpendicular to $\overleftrightarrow{AB}$. Call this line p.

There is a segment $\overline{CD}$ on $\overleftrightarrow{AB}$ such that line p is the perpendicular bisector of $\overline{CD}$. To perform the construction we will (1) locate $\overline{CD}$; (2) construct the perpendicular bisector of $\overline{CD}$.

(1) Using P as a center, and any convenient radius, construct

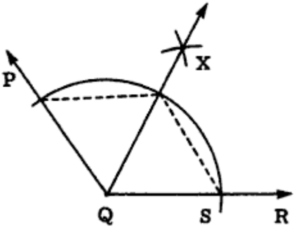
an arc which intersects $\overleftrightarrow{AB}$ at C and D . Since $\overline{PC}$ and $\overline{PD}$ are radii of the same arc, we have PC = PD . Since point P is equidistant from points C and D, point P must be on the perpendicular bisector of $\overline{CD}$.

(2) Construct the perpendicular bisector of $\overline{CD}$.

The perpendicular bisector contains point P and is perpendicular to given line $\overleftrightarrow{AB}$. Thus, the perpendicular bisector is the required line p.

• PROBLEM 9-2

Bisect a given angle.



Solution: $\overrightarrow{QX}$ bisects $\sphericalangle PQR$ if and only if $\sphericalangle PQX \cong \sphericalangle XQR$. Let T be a point on QP and S be a point on QR such that $\Delta QTX \cong \Delta QSX$. Then $\sphericalangle PQX$ and $\sphericalangle XQR$ are corresponding parts of congruent triangles and $\sphericalangle PQX \cong \sphericalangle XQR$. Therefore, if we construct ΔQTX and ΔQSX such that $\Delta QTX \cong \Delta QSX$, then $\overrightarrow{QX}$ bisects $\sphericalangle PQR$. We construct the triangles by the SSS Postulate.

Construction:

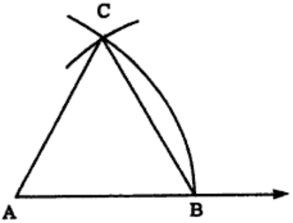
1. Using Q as the center and any radius, construct an arc of a circle that intersects $\overrightarrow{QP}$ at T and $\overrightarrow{QR}$ at S. Since radii of a circle are equal, QS = QT . By reflexivity, QX = QX .

2. Since SX must equal TX for $\Delta QTX \cong \Delta QSX$, as required by the earlier reasoning, we make certain of this by construction.
Using S as a center and a radius of more than half ST, make an arc in the interior of $\sphericalangle PQR$.
Using T as a center and the same radius, make an arc in the interior of $\sphericalangle PQR$.
Since the radii are the same, SX = TX.

3. Draw $\overrightarrow{QX}$. Since $\overline{QX} \cong \overline{QX}$, $\overline{TX} \cong \overline{SX}$, $\overline{QS} \cong \overline{QT}$, then $\Delta QTX \cong \Delta QSX$ by the SSS Postulate. Therefore, $\sphericalangle TQX \cong \sphericalangle XQS$, and $\overrightarrow{QX}$ bisects $\sphericalangle PQR$.

• PROBLEM 9-3

Construct an angle containing 60°, whose vertex is a given point. (See figure).



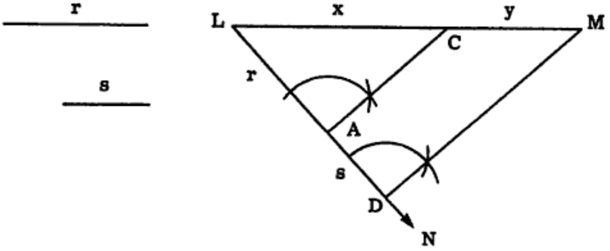
Solution: Since all angles in an equilateral triangle measure 60° (or $\frac{1}{3}(180)^\circ$), by constructing such a triangle a 60° angle can be readily determined. This may seem like a "back door" method; however, there exists a fairly straightforward way to construct an equilateral triangle. This can be accomplished by drawing two radii of a circle whose points of intersection with the circle are a linear distance apart equal to the length of the radius.

1. Given point A, construct any line segment $\overline{AB}$.
2. Using A as the center, and a radius whose length is equal to AB, construct an arc of a circle.
3. Using B as the center, and the same radius as before, mark off another arc which intersects the first arc at C.
4. Construct $\overline{CA}$ and $\overline{CB}$, forming equilateral triangle ABC. The triangle will be equilateral because all three sides are radii of arcs which have been constructed with radii of the same length.

   Since $\triangle ABC$ is equilateral, and, as such, equiangular, $m \sphericalangle CAB = \frac{1}{3}$ of 180°, the total of the measures of the angles of a triangle. Hence, $m \sphericalangle CAB = 60^\circ$, the required angle to be constructed.

• PROBLEM 9-4

Divide a given line segment into parts proportional to given line segments.



Solution: We know that a line intersecting two sides of a triangle, and parallel to the third side, will divide the two sides proportionally. First, we construct a triangle with (1) the first side equal in length to the given segment; and (2) a second side that can be easily divided into segments proportional to the given segments. Then we construct a line parallel to the third side that divides the second side into parts proportional to the given segment. By the above, we know that this line must also cut the first side, the given length, proportionally.

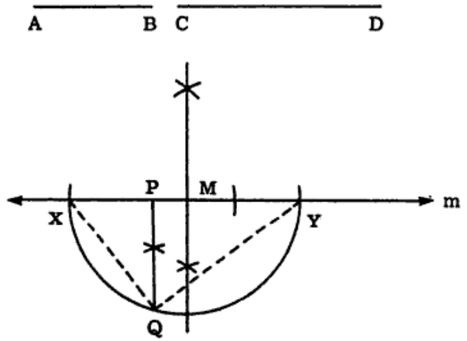
To find the second side that can be easily divided, note that there is no restriction on the length on the second side. Therefore, we construct the second side to be the length of the given segments combined.

1. Given segment $\overline{LM}$ and line segments of lengths r and s, draw $\overleftrightarrow{LN}$ making any convenient angle MLN.
2. On $\overleftrightarrow{LN}$, construct $\overline{LA}$ so that $LA = r$ and construct $\overline{AD}$ so that $AD = s$.
3. Construct $\overleftrightarrow{DM}$ to complete the triangle, ΔDML.
4. Through A, construct $\overleftrightarrow{AC}$ parallel to $\overleftrightarrow{DM}$. C is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{LM}$.
5. $\overline{LC}$ and $\overline{CM}$ are the required segments of length x and y.

The construction makes $\overleftrightarrow{AC} \parallel \overleftrightarrow{DM}$. Therefore, $x:y = r:s$ because a line parallel to one side of a triangle and intersecting the other two sides divides the other two sides proportionally.

• PROBLEM 9-5

Construct the mean proportional between two given segments.



Solution: The mean proportional of two numbers a and b is defined as that number c such that $a/c = c/b$. The key here is "mean proportional". What geometric figures have properties involving the mean proportional?

parallel.

Given: line m and point P .

Wanted: line $\overleftrightarrow{NP} \parallel m$.

Construction:

(1) Let M and Q be points on m, with Q on the right of M. Draw $\overleftrightarrow{MP}$.

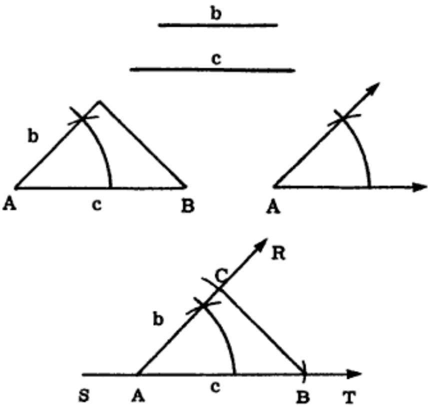
(2) Let R be any point of $\overleftrightarrow{MRP}$ on the same side of P from M. On $\overleftrightarrow{PR}$ construct $\sphericalangle$ RPN with N to the left of $\overleftrightarrow{MPR}$ such that $\sphericalangle$ RPN $\cong$ PMQ .

(3) Draw $\overleftrightarrow{NP}$. $\sphericalangle$ NPR $\cong$ $\sphericalangle$ RMQ by construction. By congruence of alternate interior angles, $\overleftrightarrow{NP} \parallel m$.

TRIANGLES

• PROBLEM 9-7

Construct a triangle when two sides and the included angle are given. (See figure).



Solution: We are given two sides of lengths b and c, as shown in the figure, and one angle, $\sphericalangle$ A. We will duplicate $\sphericalangle$ A, and then construct segments congruent to b and c along the sides of $\sphericalangle$ A. Then, by connecting the endpoints of the segments farthest away from the vertex of $\sphericalangle$ A, a unique triangle can be drawn.

Follow the construction shown in the sketch.

1. On any line $\overleftrightarrow{ST}$, use point A as the vertex to construct $\sphericalangle$ RAT $\cong$ $\sphericalangle$ A, the given angle. (The construction of an angle congruent to another angle was done previously.)

to the opposite side. In the case of an obtuse triangle, the opposite side will have to be extended, and the point of intersection between the side and the perpendicular will be outside the triangle.

The construction can be followed in the accompanying sketch:

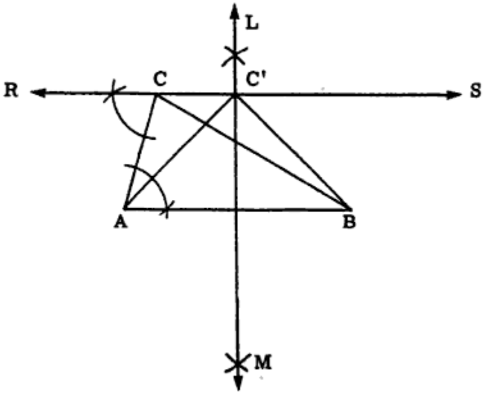
1. Through point A, construct $\overleftrightarrow{AT}$, a line perpendicular to $\overline{CB}$. Extend $\overline{CB}$ if necessary. (The construction of a perpendicular to a line through a given point outside the line has been described previously.)
2. In acute triangle ABC , $\overleftrightarrow{AT}$ intersects $\overline{CB}$ at point D. $\overline{AD}$ is the altitude from vertex A to side $\overline{CB}$.
3. In obtuse triangle ABC, $\overleftrightarrow{AT}$ intersects $\overline{CB}$ extended at point D. Line segment $\overline{AD}$ is the altitude from vertex A to side $\overline{CB}$.
4. In right triangle ABC, the point of intersection of $\overleftrightarrow{AT}$ and side $\overline{BC}$ is a vertex of the triangle, namely vertex B. Therefore, in addition to being one side of the triangle, $\overline{BC}$ is the altitude of right triangle ABC. This is so because $\measuredangle$ B is a right angle.

• PROBLEM 9-9

Construct an isosceles triangle which has the same base as a given scalene triangle, and which is equal in area to it. (See figure).

Solution: The key to this construction lies in the fact that the area of a triangle is equal to $\frac{1}{2}$ x base x height. The isosceles triangle to be constructed is to have the same base as the given scalene triangle. To insure that the isosceles triangle has the same area as the scalene triangle, we must construct it in such a way that both triangles have the same altitude.

If we draw line $\overleftrightarrow{RS}$ parallel to base $\overline{AB}$, the distance between the two lines will be the altitude of the scalene triangle (ΔABC).



Any other perpendicular distance between $\overleftrightarrow{RS}$ and $\overline{AB}$ will be equal to the altitude of the scalene triangle. The altitude of the isosceles chord in the circle. By choosing two non-parallel chords (so that their perpendicular bisectors do not coincide) and constructing their perpendicular bisectors, we know that only the point that lies on both bisectors can be the center.

Given: Any circle.

Wanted: The center, O, of the circle.

Construction:

(1) Draw any two nonparallel chords and label them $\overline{AB}$ and $\overline{CD}$.

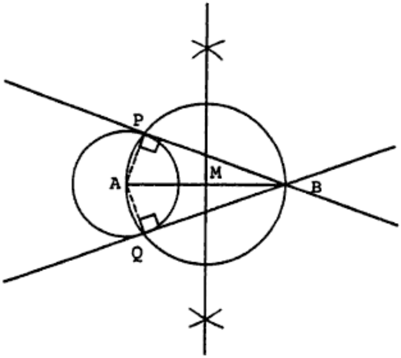
(2) Construct the perpendicular bisector of each chord. Name them $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$, respectively.

(3) The intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ is O, the center of the circle.

• PROBLEM 9-11

Construct the lines tangent to a circle through a point external to the circle.

Solution: Let P be the point of tangency. A line is tangent to a circle at point X, if the line is perpendicular to the radius drawn to point X. Therefore, in the figure shown, $\overline{BP} \perp \overline{PA}$ or $\measuredangle BPA$ is a right angle and ΔBPA is a right triangle. Thus, there are two requirements for point P: (i) P, the point of tangency, must be on circle A ; and (ii) ΔBPA must be a right triangle. The locus of points satisfying condition (i) is the circle A. The locus of points satisfying condition (ii) is the circle (not the semicircle) with $\overline{BA}$ as the diameter. (To see this, remember that any triangle inscribed in a semicircle is a right triangle with the hypotenuse as the diameter. The converse is also true. All right triangles with given hypotenuse length can be inscribed in a semicircle with diameter equal to the hypotenuse.)



The intersection of the loci are thus the possible points of tangency, point P and Q .

Given: Point B in the exterior of ⊙A .

Wanted: The lines through B tangent to ⊙A .

Construction:

(1) Draw and bisect $\overline{AB}$. Label the midpoint M .

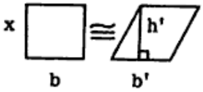
(2) With M as center and MB as the radius construct the circle through A and B. Circle M, with diameter $\overline{AB}$, intersects ⊙A at P and Q .

(3) Draw $\overrightarrow{BP}$ and $\overrightarrow{BQ}$, the required tangents.

POLYGONS

• PROBLEM 9-12

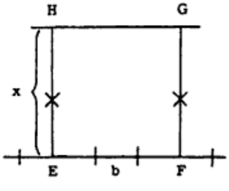
Construct a rectangle that has a given base length and is equal in area to the area of a given parallelogram. (See figure).

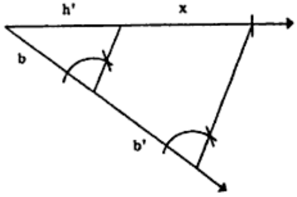


Area of Rectangle = Area of Parallogram

$$bx = b'h'$$

$$b : b' = h' : x$$





Solution: We are given parallelogram ABCD, with base of length b' and altitude of length h' . Accordingly, area of ▱ ABCD = b'h' .

The rectangle we want to construct has a known base length of b, and an unknown altitude, say x. Then, area of rectangle to be constructed = bx .

Since we want the area of the rectangle to equal the area of the parallelogram, it must be that bx = b'h' . Therefore,

$$\frac{b}{b'} = \frac{h'}{x}$$

or b: b' = h':x is a proportion. We must find the fourth proportional of b,b' and h, (i.e., x), and then construct the rectangle with base b and altitude x.

Follow the construction in the accompanying sketch:

(1) Construct $\overline{DR}$, the altitude of ▱ABCD, whose length has been represented by h' .

(2) Construct the fourth proportional to b,b', and h' , (i.e., x). (This construction is vitally important to the problem. It was done previously.)

(3) The segment we have just determined, whose length is x, is congruent to the altitude of the required rectangle.

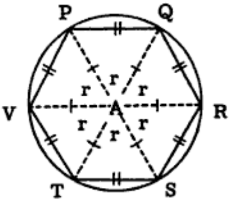
(4) Let $\overline{EF}$ represent the base of the rectangle of length b. At points E and F, construct perpendiculars to $\overline{EF}$ of length x; these are the altitudes of the rectangle. (This construction has also appeared previously.)

(5) Through points H and G, construct $\overline{HG}$. Since $\overline{HE} \cong \overline{FG}$, $\overline{HE} \parallel \overline{FG}$ and ∡ HEF is a right angle. EFGH is the required rectangle, whose base is a segment of length b, and whose area is equal to the area of ▱ABCD .

Since the construction makes x the fourth proportional of b,b' and h' , then b : b' = h': x or bx = b'h' . Since the area of ▱ABCD = b'h' , and the area of rectangle EFHG = bx , the area of rectangle EFGH = area of ▱ABCD. Therefore, EFGH is the rectangle we set out to construct.

• PROBLEM 9-13

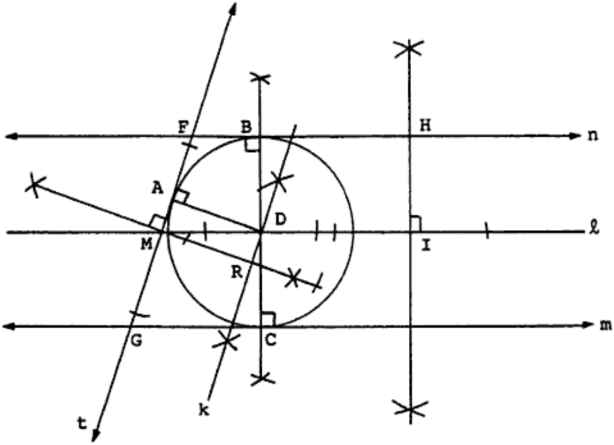
Inscribe a regular hexagon in a circle.



Solution: Consider the n-sided regular polygon in the figure above (n, in this case, equals 6). From the center of the circumscribed circle, draw the radii to the vertices. Because all radii are congruent, two sides of every triangle must be congruent to two sides of every other triangle. Because this is a regular polygon, all the sides, $\overline{VT}$, $\overline{VP}$, $\overline{PQ}$, etc., must be congruent. By the SSS Postulate, all the triangles must be congruent, and, by corresponding parts, all the central angles must be congruent. Since the central angles must sum to 360° , the measure of the central angle of each side of an n-sided regular polygon must be 360/n .

This is the crux of every problem that involves inscribing a reg-

Given two parallel lines and a transversal $\overleftrightarrow{FG}$, construct the circle tangent to both the parallel lines and the transversal.



Solution: To construct a circle, we must find (a) the center; and (b) the radius.

(a) The center: Consider the radii drawn to the tangent points, $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$. Because these are radii drawn to tangent points, $\overline{AD} \perp$ line t, $\overline{BD} \perp$ line n, and $\overline{CD} \perp$ line m. Then, by definition, AD, BD, and CD are the distances from lines t,n, and m to point D. Therefore, D, the center of the circle, which must be equidistant from t,m, and n.

The locus of points that are equidistant from n and m is the line parallel to n and m and equidistant from both. Find the midpoint of the transversal segment cut by n and m and construct line ℓ parallel to n. The distance BD between n and ℓ must be the radius of the circle. The center must also be a distance HI from t. The locus of points a distance HI from t is two parallel lines a distance HI from t . The center of the circle is the intersection of ℓ and the lines parallel to t.

(b) The radius is the length of $\overline{HI}$ where $\overleftrightarrow{BD}$ is a line perpendicular to n,m, and ℓ .

Given: Two parallel lines n and m, and a transversal t. Let F and G be the points of intersection of the transversal with n and m.

Wanted: Construct the circle tangent to both the parallel lines and the transversal.

Construction:

(1) Locate the midpoint of $\overline{FG}$, label it M.

(2) Construct line ℓ through M parallel to n.

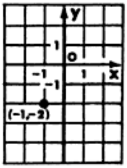
CHAPTER 10

COORDINATE GEOMETRY

PLOTTING POINTS

• PROBLEM 10-1

Describe the location of the point (-1,-2) and then plot it on coordinate graph paper.



Solution: The first number in the point, the abscissa, is -1 and represents the number of horizontal units along the x-axis that will be moved left of the origin of the Cartesian plane when plotting this point. The second number, the ordinate, is -2 which represents the vertical distance to be moved downward from the origin.

With this in mind, we can plot the required point on the graph. We find that (-1,-2) will lie in the lower left region of the graph, Quadrant III, 2 units below the x-axis and 1 unit to the left of the y-axis.

• PROBLEM 10-2

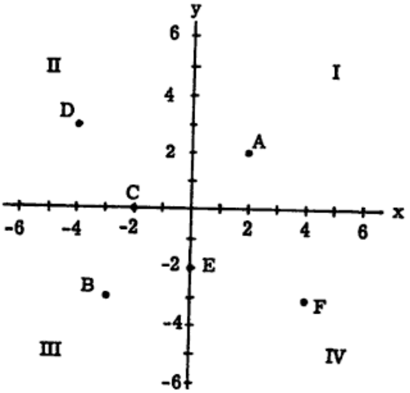
Give the coordinates of points A,B,C,D,E,and F in the figure below and determine the quadrants in which they lie.

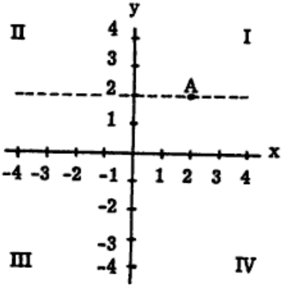
Solution: To find the coordinates of a point, z for example, imagine a line pa-allel to the y axis passing through z.

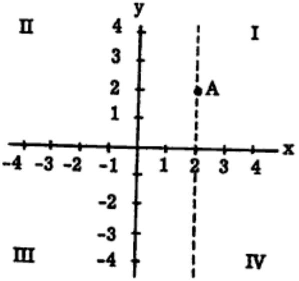
Where that line intersects the x-axis, gives the x-coordinate. Similarly, the imaginary line parallel to the x-axis passing through z, gives the y coordinate. For example, the y-coordinate of A is 2. Similarly, the x-coordinate of A is 2.

Repeating the procedure for points B,C,D,E, and F, we obtain the table below.

Point	Coordinate	Quadrant
A	(2, 2)	I
B	(-3, -3)	III
C	(-2, 0)	on the x-axis
D	(-4, 3)	II
E	(0, -2)	on the y-axis
F	(4, -3)	IV







In determining the quadrant of a point, note that any point above the x-axis and to the right of the y-axis is in the first quadrant. A more common way of expressing first quadrant is to use the Roman numeral notation. Thus the first quadrant becomes Quadrant I. Similarly, if the point is above the x-axis but to the left of the y-axis, it is said to be in the second quadrant, or Quadrant II. Points below the x-axis and to the left of the y-axis are in Quadrant III; and points below the x-axis but to the right of the y-axis are in Quadrant IV. Points that lie on the x or y axis are said to lie on the x or y axis and are not in any quadrant.

There is a second method of determining the quadrant of the point. Note that all points above the x-axis have positive y-coordinates. Similarly, all points below the x-axis have negative y-coordinates. Points to the right of the y-axis have positive x-coordinates. Points to the left of the y-axis have negative x-coordinates. Since all points in Quadrant I are above the x-axis and to the right of the y-axis, all points in Quadrant I have the form (positive, positive). Repeating the procedure for the other five cases, we obtain:

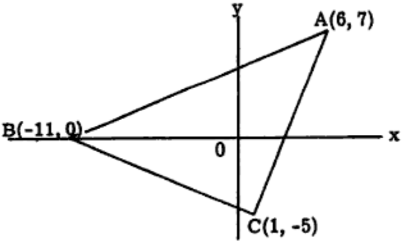
Quadrant II: (Negative, positive)
Quadrant III: (negative, negative)
Quadrant IV: (positive, negative)
On the x-axis: (positive or negative, 0)
On the y-axis: (0, positive or negative).

Given the graph and asked to find the quadrant of the point, method 1 is easier. If we are given the coordinates, method 2 is better.

DETERMINING DISTANCE

• PROBLEM 10-3

Problem: Show that the triangle with vertices A(6,7), B(-11,0), and C(1,-5) is isosceles.



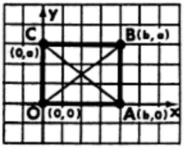
Solution: To prove that triangle ABC is isosceles, we must find two equal sides. Calculate the length of the sides using

$$= \sqrt{1 + 9} = \sqrt{10}$$

The length of the radius is $\sqrt{10}$.

• PROBLEM 10-5

Prove, using coordinate geometry, that the diagonals of a rectangle are congruent.



Solution: Prior to getting into the proof, let us draw a rectangle on the Cartesian plane. (See figure). Place one vertex at the origin and side $\overline{OA}$ along the x-axis, as shown. Extend side $\overline{AB}$ into quadrant I, so that vertex B is the point (b,a). The coordinates of the vertices of the rectangle are O(0,0), A(b,0), B(b,a), and C(0,a).

Since ABCO is given to be a rectangle, with $\overline{OB}$ and $\overline{CA}$ as the diagonals, proving OB = CA will be sufficient to prove that the diagonals of a rectangle are congruent.

We can use the general formula for the distance between two points, $d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$, to obtain the desired results. Here, (x_1,y_1) and (x_2,y_2) are the coordinates of the 2 points. Distance OB is given by

$$OB = \sqrt{(0-b)^2 + (0-a)^2} = \sqrt{b^2 + a^2}.$$

Distance AC is given by

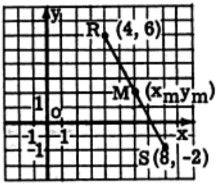
$$AC = \sqrt{(b-0)^2 + (0-a)^2} = \sqrt{b^2 + a^2}.$$

Therefore, since OB = AC we can conclude that $\overline{OB} \cong \overline{AC}$.

MIDPOINTS

• PROBLEM 10-6

Find the coordinates of the midpoint of the line segment which joins the point R(4,6) and the point S(8,-2). (See figure)



Solution: By a theorem, we know that the coordinates of the midpoint of a line segment are equal to one-half the sum of the corresponding x and y coordinates of the endpoints of the line segment. (These results can be checked by applying the general formula for the distance between two points). Let $M = (x_m, y_m)$, the midpoint of segment $\overline{RS}$.

Also let $R = (x_1, y_1)$ where $x_1 = 4$, $y_1 = 6$

and $S = (x_2, y_2)$ where $x_2 = 8$, $y_2 = -2$.

Then, $x_m = \frac{1}{2}(x_1+x_2) = \frac{1}{2}(4+8) = \frac{1}{2}(12) = 6$

and $y_m = \frac{1}{2}(y_1+y_2) = \frac{1}{2}(6-2) = \frac{1}{2}(4) = 2.$

To check that $M = (6,2)$ is the midpoint of $\overline{RS}$, calculate RM and MS by using the general distance formula to verify that they are equal.

$$RM = \sqrt{(x_1-x_m)^2 + (y_1-y_m)^2}$$

By substitution, $RM = \sqrt{(4-6)^2(6-2)^2} = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16}$

$$RM = \sqrt{20}$$

$$MS = \sqrt{(x_m-x_2)^2 + (y_m-y_2)^2}$$

By substitution, $MS = \sqrt{(6-8)^2 + (2-(-2)^2} = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16}$

$$MS = \sqrt{20}$$

Therefore, RM = MS

Therefore, the coordinates of the midpoint of line segment $\overline{RS}$ are $x = 6$, $y = 2$ or $(6,2)$.

• PROBLEM 10-7

The midpoints of the sides of a triangle are (2,5), (4,2), (1,1). Find the coordinates of its three vertices.

$$\frac{y_1+y_3}{2} = 5, \quad y_1+y_3 = 10 \qquad (4)$$

$$\frac{y_1+y_2}{2} = 2, \quad y_1+y_2 = 4 \qquad (5)$$

$$\frac{y_2+y_3}{2} = 1, \quad y_2+y_3 = 2 \qquad (6)$$

Again sum (4) and (5) to obtain $2y_1+y_2+y_3 = 14$

But (6) tells us $y_3+y_2 = 2$. Hence, by substitution, we have

$$2y_1+2 = 14 \text{ and } y_1 = 6.$$

From (4), $y_3 = 10-y_1 = 10-6 = 4 = y_3$

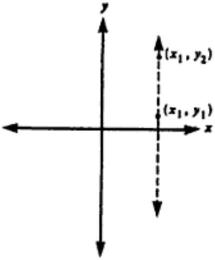
From (5), $y_2 = 4-y_1 = 4-6 = -2 = y_2$

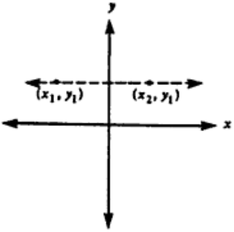
The vertices of the triangle in this problem are A = (5,6), B = (3,-2), and C = (-1,4).

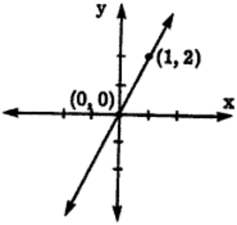
SLOPES

• PROBLEM 10-8

What is the slope of the line that passes through the origin and point (1,2)? the slope of a horizontal line? a vertical line?







Solution: The slope of the line determined by two points, (x_1,y_1) and (x_2,y_2) such that $x_1 \neq x_2$ and is the same as m, the slope of any segment of the line. The line that passes through the origin and point (1,2) is determined by the points (0,0) and (1,2), Therefore, the slope of the line is the same as m, the slope of the line segment between (0,0) and (1,2).

$$\text{(i)} \quad m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-0}{1-0} = 2 .$$

A horizontal line is, by definition, parallel to the x-axis. Therefore all the ordinates must be equal. Let (x_1,y_1) and (x_2,y_2) be two distinct points on the horizontal line. Then the slope of the line equals

$$\text{(ii)}\quad m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

Since (x_1,y_1) and (x_2,y_2) lie on the horizontal line, $y_2 = y_1$. Therefore, the numerator $y_2 - y_1 = 0$, and $m = 0$. The slope of a horizontal line equals zero.

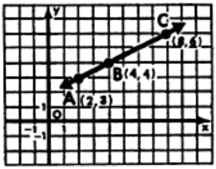
A vertical line is, by definition, parallel to the y-axis, and, consequently, all abscissas are equal: For all points (x_1,y_1) and (x_2,y_2) on the line, $\underline{x_1 = x_2}$.

Therefore, we cannot apply the above theorem since the theorem requires that the line be determined by two points $(_1,y_1)$ and (x_2,y_2) such that $\underline{x_1 \neq x_2}$. No two such points exist on the vertical line and we are left to conclude that a vertical line has no slope.

If we did try to find the slope using the formula, then we would obtain $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. For a vertical line, $x_2 = x_1$; thus $x_2 - x_1 = 0$ and $m = \frac{y_2 - y_1}{0}$. Because the denominator is zero, the value of m is not defined.

• PROBLEM 10-9

Prove that points A(2,3), B(4,4), and C(8,6) are collinear.



Solution: Three points are collinear if the slope of the line segment between the first and second point is equal to the slope of the line segment between the second and third points.

Let $A = (x_1,y_1) = (2,3)$

$B = (x_2,y_2) = (4,4)$

$C = (x_3,y_3) = (8,6)$

These three points have been plotted on the accompanying graph. If the slope of $\overline{AB}$ equals the slope of $\overline{BC}$, the points are collinear.

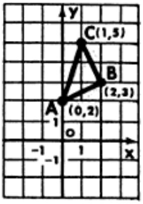
$$\text{Slope of } \overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3}{4 - 2} = \frac{1}{2}$$

$$\text{Slope of } \overline{BC} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{6 - 4}{8 - 4} = \frac{2}{4} = \frac{1}{2}$$

Since the slope of $\overline{AB}$ = slope of $\overline{BC}$, the points A,B, and C all lie on the same straight line, $\overleftrightarrow{AC}$, and are, therefore, collinear.

• PROBLEM 10-10

Prove, by means of slope, that the triangle plotted in the accompanying graph, whose vertices are A(0,2), B(2,3), and C(1,5), is a right triangle.



Solution: We can prove that lines $\overline{CB} \perp \overline{AB}$, making ΔABC a right triangle, by demonstrating that the slope of $\overline{CB}$ is the reciprocal of the slope of $\overline{AB}$.

Let $A = (x_1,y_1) = (0,2)$

$B = (x_2,y_2) = (2,3)$

$C = (x_3,y_3) = (1,5)$,

then,

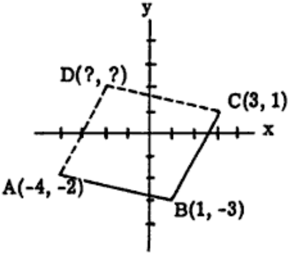
$$\text{Slope of } \overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 2}{2 - 0} = \frac{1}{2}$$

$$\text{Slope of } \overline{CB} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{5 - 3}{1 - 2} = -\frac{2}{1}$$

The slope of $\overline{AB}$ is the negative reciprocal of the slope of $\overline{CB}$ because $(\frac{1}{2})(-2) = 1$. Therefore, $\overline{AB} \perp \overline{BC}$. Hence, ΔABC is a right triangle because it contains a right angle.

• PROBLEM 10-11

Given A(-4,-2), B(1,-3), and C(3,1) find the coordinates of D, in the 2nd quadrant such that quadrilateral ABCD is a parallelogram.



Solution: The many properties of a parallelogram allow us several methods of proceeding; however, not all of them are equally straightforward. We know, for example, that opposite sides of a parallelogram are congruent and, therefore, AD = CB and DC = AB. However, to use the distance formula

$$d = \sqrt{(x_1-x_2)^2 + (y_2-y_1)^2}$$

would yield two equations with square roots of unknowns. Similarly, any attempt to show congruence of angle would bring more complications than it would solve. The best method is to use the property which states that opposite sides of a parallelogram are parallel. In coordinate geometry, this reduces down to the problem of showing that the slopes of the opposite sides are equal. Let the coordinates of D be (x,y)

Slope of $\overline{AB} = \frac{-2-(-3)}{-4-1} = -\frac{1}{5}$

Slope of $\overline{DC} = \frac{y-1}{x-3}$

Slope of $\overline{CB} = \frac{1-(-3)}{3-1} = \frac{4}{2} = 2$

Slope of $\overline{DA} = \frac{y-(-2)}{x-(-4)} = \frac{y+2}{x+4}$

By the properties of the parallelogram, $\overline{AB} \parallel \overline{DC}$

(i) slope of $\overline{AB}$ = slope of $\overline{DC}$

(ii) $$-\frac{1}{5} = \frac{y-1}{x-3}$$

By crossmultiplication,

(iii) $x - 3 = -5(y-1) = -5y + 5$

(iv) $x + 5y - 3 = 5$

(v) $x + 5y = 8$

Since $\overline{DA}$ and $\overline{CB}$ are also opposite sides,

(vi) slope of $\overline{DA}$ = slope of $\overline{BC}$

(vii) $2 = \frac{y + 2}{x + 4}$

(viii) $y + 2 = 2(x+4) = 2x + 8$

(ix) $-2x + y + 2 = 8$

(x) $-2x + y = 6$

Equations (v) and (x) form a system of two simultaneous equations in two unknowns. We can eliminate x from the equation by multiplying equation (v) by 2 and adding the result to equation (x):

$$\begin{array}{lr} \text{(xi)} & 2x + 10y = 16 \\ \text{(x)} & \underline{+\ -2x + \quad y = \ \ 6} \\ \text{(xii)} & 11y = 22 \\ \text{(xiii)} & y = \frac{1}{11} \cdot 22 = 2 \end{array}$$

Substitution y = 2 into equation (v), we obtain:

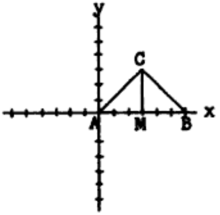
(xiv) $x + 5(2) = 8$

(xv) $x = 8 - 5(2) = 8 - 10 = -2.$

Therefore, D = (-2,2) for ABCD to be a parallelogram.

• PROBLEM 10-12

Given: A(0,0), B(6,0), and C(3,3), find the equation for the median to side $\overline{AB}$.



Solution: To find the equation of a line, we need two points on the line. Let $\overline{CM}$ be the median to $\overline{AB}$. Then vertex C is one point on the median. Similarly M, the midpoint of $\overline{AB}$, is also on the median. Using the Midpoint Formula,

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{0+6}{2}, \frac{0+0}{2}\right) = (3,0).$$

Thus, with M(3,0), C(3,3), the equation of the median is given by the Point-slope form.

Form: $y-y_1 = m(x-x_1)$, where (x_1,y_1) is point $M(3,0)$ and the slope $m = \frac{\Delta y}{\Delta x} = \frac{3-0}{3-3} = \frac{3}{0}$ or undefined.

The only straight lines whose slopes are undefined are vertical lines. The equation of a vertical line is $x=c$, where c is the abscissa of any point on the line. Since the slope of $\overleftrightarrow{CM}$ is undefined, $\overleftrightarrow{CM}$ is a vertical line containing points (3,3) and (3,0), 3 is the common abscissa. Therefore, the equation of $\overleftrightarrow{CM}$ is $x=3$.

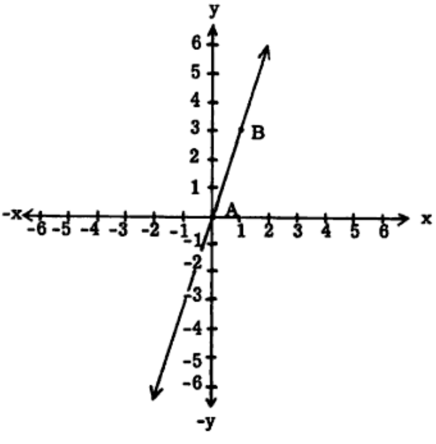
There is one last step. $x=3$ is an equation of a line. A median is a line segment. Therefore, we must make it clear that the endpoints are (3,3) and (3,0) - that the ordinates must be between 3 and 0. Equation of the median:

$$x=3 \text{ where } 0 \le y \le 3.$$

LINEAR EQUATIONS

• PROBLEM 10-13

a) Draw a line with slope 3 through the origin. b) Write the coordinates of the points whose x coordinates are -2, -1, 0, 1, 2, 3. c) Write the equation of the line. d) Describe the set of coordinates of the line two ways using set builder notation. e) Verify that the coordinates of part (b) satisfy the equation of part (c).



two alternative ways. First, the line AB consists of all coordinate pairs (x,y) such that x is an element of the set of real numbers, and y is 3x. That is,

$$S = \{(x,y) \mid x \in R,\ y = 3x\},$$

where we have labelled the set of points lying on $\overleftrightarrow{AB}$ as S. Alternatively, the line $\overleftrightarrow{AB}$ consists of all coordinate pairs (x,3x) such that x is an element of the set of real numbers (Here we have used the fact that y = 3x). In set builder notation,

$$S = \{(x,3x) \mid x \in R\}.$$

e) By the table of part (b) and equation (1), all the coordinates of part (b) satisfy the equation of part (c).

• PROBLEM 10-14

Write an equation of the line which is parallel to 6x + 3y = 4, and whose y-intercept is -6.

Solution: We employ the slope intercept form for the equation to be written, since we are given the y-intercept. Our task is then to determine the slope.

We are given the equation of a line parallel to the line whose equation we wish to find. We also know that the slopes of two parallel lines are equal. Hence, by finding the slope of the given line, we will also be finding the unknown slope. To find the slope of the given equation 6x + 3y = 4, we transform the equation 6x + 3y = 4 into slope intercept form.

$$6x + 3y = 4$$

$$3y = -6x + 4$$

$$y = -\frac{6}{3}x + \frac{4}{3}$$

$$y = -2x + \frac{4}{3}.$$

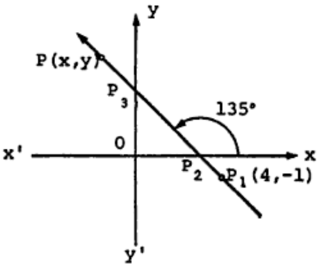
Therefore, the slope of the line we are looking for is -2. The y-intercept is -6. Applying the slope intercept form, y = mx + b, to the unknown line, we obtain,

$$y = -2x - 6$$

as the equation of the line.

• PROBLEM 10-15

Find the equation of the straight line passing through the point (4,-1) and having an angle of inclination of 135°.



Solution: We are given one point on the line. If we had the slope of the line, we could apply the point slope theorem to determine the equation.

To determine the slope we will make use of the fact that the angle of inclination is 135°. The angle of inclination is the angle between the positive x-axis and the part of the line above the x-axis with the interior of the angle lying to the right of the line.

We shall show that the slope of the line = $- \tan \sphericalangle P_3P_2O$ = tan (angle of inclination).

Given points $P_2(x',0)$ and $P_3(0,y')$ on the line we can represent the slope m as

$$m = \frac{y' - 0}{0 - x'} = -\frac{y'}{x'}$$

In right triangle P_3OP_2, y' and x' are the respective lengths of sides $\overline{OP_3}$ and $\overline{OP_2}$. Hence, the tangent of $<P_3P_2O$ is the

$$\frac{\text{length of leg opposite } <P_3P_2O}{\text{length of leg adjacent } <P_3P_2O} \quad \text{or}$$

$$\tan \sphericalangle P_3P_2O = \frac{y'}{x'}$$

Since $m = -\frac{y'}{x'}$ and

$$\tan \sphericalangle P_3P_2O = \frac{y'}{x'} \,,$$

by substitution,

$$m = -\tan \sphericalangle P_3P_2O$$

Exterior angle $<P_3P_2X$ is given as 135° and can be observed to be the supplement of $\sphericalangle P_3P_2O$. Accordingly, $m\sphericalangle P_3P_2O = 180° - 135° = 45°$. Hence, by substitution, $m = -\tan 45° = 1$. Now

we know a point on the line, (4,-1) and the slope m = -1. The point - slope theorem is $(y-y_1) = m(x-x_1)$. Making one final substitution, we obtain

$$(y+1) = -1(x-4)$$

$$y+1 = -x + 4$$

$$y = -x + 3$$

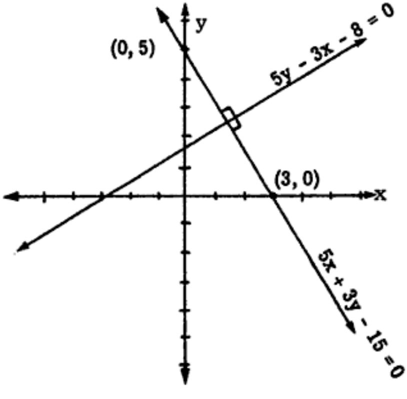
Therefore, the equation of the line is given by $y = -x + 3$ or $y + x = 3$.

• PROBLEM 10-16

Find the equation of the perpendicular bisector of that portion of the straight line $5x + 3y - 15 = 0$ which is intercepted by the coordinate axes.

Solution: Prior to finding the midpoint of the segment and the negative reciprocal of its slope, which are essential to determining the equation of the perpendicular bisector, we must more clearly describe the segment.

To determine the slope of the segment, let us rearrange the equation into the form $y = mx + b$, where m is the slope and b is the y-intercept.



$$5x + 3y - 15 = 0$$
$$3y = -5x + 15$$
$$y = -\frac{5}{3}x + 5.$$

Hence, the slope, m, is $-\frac{5}{3}$. The coordinate axes provide the boundary of the segment in question. Since the y-intercept is 5, one endpoint is (0,5).

The other endpoint can be found by setting $y = 0$ and solving for x.

$$\begin{aligned} 0 &= -\frac{5}{3}x + 5 \\ -5 &= -\frac{5}{3}x \\ 5\left(\frac{3}{5}\right) &= x \\ 3 &= x. \end{aligned}$$

Hence, the other endpoint is (3,0).

The perpendicular bisector of the segment will have slope $-\frac{1}{m}$, or $\frac{3}{5}$. Also, it will pass through the midpoint of the given segment, $\left(\frac{0+3}{2}, \frac{5+0}{2}\right)$ or $\left(\frac{3}{2}, \frac{5}{2}\right)$.

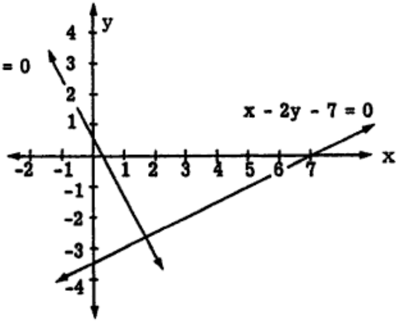
We apply the point-slope form, $(y-y_1) = m(x-x_1)$, to obtain the equation of the perpendicular bisector. Hence,

$$\begin{aligned} \left(y - \frac{5}{2}\right) &= \frac{3}{5}\left(x - \frac{3}{2}\right) \\ y - \frac{5}{2} &= \frac{3}{5}x - \frac{9}{10} \\ y &= \frac{3}{5}x + \frac{16}{10}. \end{aligned}$$

Multiplying through by 5 to remove the fractions, we obtain, for the equation of perpendicular bisector, $5y - 3x - 8 = 0$.

• PROBLEM 10-17

Problem: Find the point of intersection of the two lines, $4x + 2y - 1 = 0$, $x - 2y - 7 = 0$.



Solution: If $P_0(x_0,y_0)$ is the required point, and if we use the two slope intercept forms of these equations, we see that x_0 and y_0 must satisfy them both. The slope intercept form of $4x + 2y - 1 = 0$ is $y = -2x + \frac{1}{2}$ (just solve for y). Similarly, the slope intercept form of $x - 2y - 7 = 0$ is $y = \frac{1}{2}x - \frac{7}{2}$. Plugging in (x_0,y_0), the points to be determined, yields two expressions for y_0,

$$y_0 = -2x_0 + \frac{1}{2} \quad \text{and} \quad y_0 = \frac{1}{2}x_0 - \frac{7}{2}.$$

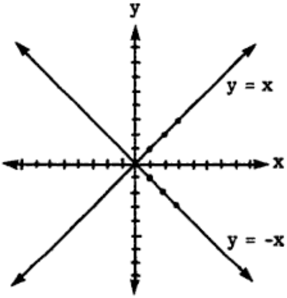
Substituting the second value for y_0 back into the first equation gives us an expression in x_0 alone:

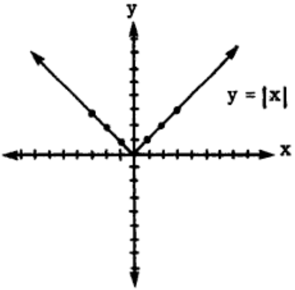
$$\frac{1}{2}x_0 - \frac{7}{2} = -2x_0 + \frac{1}{2}.$$

Collecting like terms yields $\frac{5}{2}x_0 = \frac{8}{2}$, or $x_0 = \frac{8}{5}$. Substituting this value into either of the slope-intercept forms gives $y_0 = -\frac{27}{10}$. Thus, the point of intersection is $(\frac{8}{5}, -\frac{27}{10})$.

• PROBLEM 10-18

a) Draw the graphs of $S = \{(x,y) \mid y = x\}$ and $T = \{(x,y) \mid y = -x\}$.
b) Draw the graph of $U = \{(x,y) \mid y = |x|\}$.





Solution: a) The graph of the set S consists of all points (x,y) such that $y = x$. The points (0,0), (1,1), (2,2) and (3,3) all satisfy this requirement. The graph, a straight line, is shown in figure (a).

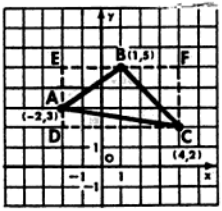
The graph of the set T consists of all points (x,y) such that $y = -x$. The point (0,0), (1,-1), (2,-2) and (3,-3) satisfy this requirement. The graph, a straight line, is shown in figure (a).

Note that three points were used but two were sufficient to determine the line.

AREAS

• PROBLEM 10-20

Plot the points A (- 2, 3), B (1, 5) and C (4, 2) and find the area of ΔABC.



Solution: After plotting the points, as the graph shows, we find that no side is parallel to either of the axes. Therefore, when the altitude is drawn, its length is not easily determined. This makes the standard area formula for a triangle inapplicable.

To overcome this problem, we can inscribe the triangle in a figure whose sides are parallel to the axis, the lengths easier to calculate, and whose area formula can be applied. Any area within this figure but outside of the figure of main concern must be subtracted from the larger area in order to obtain the desired results.

In this problem, construct rectangle EFCD by drawing lines parallel to the x-axis through B and C and lines parallel to the y-axis through A and C.

The area of ΔABC = area of rectangle EFCD - sum of the areas of right triangles, CDA, BEA and BFC.

Now, the endpoints of all line segments, whose lengths are needed for the calculation, will match in either the x or y coordinates. The length will be equal to the absolute value of the difference of the non-matching coordinates of the two endpoints for each segment. We will calculate the relevant lengths and then the area.

Since, $A = (-2, 3)$, $B = (1, 5)$, $C = (4, 2)$

$$D = (-2, 2),\ E = (-2, 5),\ F = (4, 5)$$

Therefore,

$$ED = |5 - 2| = 3$$

$$EF = |4 - (-2)| = 6$$

$$EA = |5 - 3| = 2$$

$$EB = |1 - (-2)| = 3$$

$$BF = |4 - 1| = 3$$

$$FC = |5 - 2| = 3$$

$$DC = |4 - (-2)| = 6$$

$$AD = |3 - 2| = 1$$

Recall,

Area of ΔABC = area of rectangle EFCD - (area of rt. ΔCDA + area of rt. ΔBEA + area of rt. ΔBFC).

We must now determine the areas on the right side then substitute for the desired results.

Area of rectangle EFCD = bh = DC × ED = 6 × 3 = 18

Area of rt. ΔCDA = $\frac{1}{2}$ leg × leg = $\frac{1}{2}(AD \times DC) = \frac{1}{2}(1 \times 6) = 3$

Area of rt. ΔBEA = $\frac{1}{2}$ leg × leg = $\frac{1}{2}(EB \times EA) = \frac{1}{2}(2 \times 3) = 3$

Area of rt. ΔBFC = $\frac{1}{2}$ leg × leg = $\frac{1}{2}(BF \times FC) = \frac{1}{2}(3 \times 3) = 4.5$

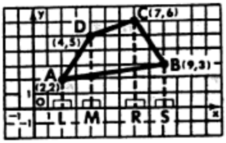
Therefore, by substitution,

Area of $\Delta ABC = 18 - (3 + 3 + 4.5) = 18 - 10.5$

Therefore, Area of ΔABC = 7.5 sq. units.

• PROBLEM 10-21

Find the area of the polygon whose vertices are A (2, 2), B (9, 3), C (7, 6), and D (4, 5).



Solution: The polygon described in this question, and drawn on the accompanying graph, does not have any opposite sides parallel to each other, in which case it cannot be a type of parallelogram or trapezoid. Additionally, it is not regular, since no two sides are congruent. Therefore, no standard area formula for quadrilaterals can be applied.

We must therefore use a method for solution in which figures are constructed which have straightforward formulas for area and the length of whose sides and altitudes are known. This will enable us to indirectly determine the area of the given figure.

The usual approach is to extend lines from the vertices, perpendicular to one of the axis and calculate the area of the largest figure bounded by these lines and the sides of the polygon. We then subtract any area lying in this region not in the polygon, whose area we desire to find.

In this problem we draw line $\overleftrightarrow{AL}$, $\overleftrightarrow{DM}$, $\overleftrightarrow{CR}$ and $\overleftrightarrow{BS}$ all perpendicular to the x-axis. Since they are all perpendicular to the same line, they are all parallel to each other and, as such, will provide the bases of several trapezoids that will be used in this calculation. The trapezoids are LMDA, MRCD, RSBC and LSBA.

The area of polygon ABCD can be found by adding the areas of trapezoids LMDA, MRCD, RSBC and subtracting the area of trapezoid LSBA from the sum.

The formula for the area of a trapezoid with altitude h and bases b_1 and b_2 is

$$A = \tfrac{1}{2}\, h\, (b_1 + b_2).$$

Area of ABCD = (area of trapezoid LMDA + area of trapezoid MRCD + area of trapezoid RSBC) - area of trapezoid LSBA.

We will calculate the area of each trapezoid and substitute it into the above to find the area of **ABCD.**

The lengths of the bases of the trapezoids will be given as the distance from each vertex to the x-axis; and the altitudes as the horizontal distance between the bases along the x-axis.

Let us determine all necessary base and altitude lengths. The base lengths will equal the y-component of the vertex from which it is drawn, since that represents the vertical distance the vertex is above the x-axis.

Therefore, $AL = 2$, $DM = 5$, $CR = 6$, and $BS = 3$.

The altitude lengths will be given by the absolute value of the difference between the x-components of the two "vertex points" in each trapezoid.

Therefore, $LM = |4 - 2| = 2$

$MR = |4 - 7| = 3$

$RS = |9 - 7| = 2$

$LS = |2 - 9| = 7$

Accordingly,

Area of trapezoid LMDA $= \tfrac{1}{2}(LM)(AL + DM) = \tfrac{1}{2}\,(2)(2 + 5)$

$= \frac{1}{2}(2)(7) = 7.$

Area of trapezoid MRCD $= \frac{1}{2}(MR)(DM + CR) = \frac{1}{2}(3)(5 + 6)$

$= \frac{1}{2}(3)(11) = 16.5$

Area of trapezoid RSBC $= \frac{1}{2}(RS)(CR + BS) = \frac{1}{2}(2)(6 + 3)$

$= \frac{1}{2}(2)(9) = 9$

Area of trapezoid LSBA $= \frac{1}{2}(LS)(AL + BS) = \frac{1}{2}(7)(2 + 3)$

$= \frac{1}{2}(7)(5) = 17.5$

By substitution,

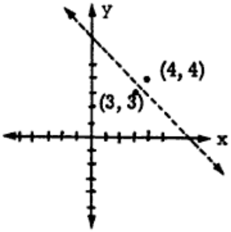
Area of ABCD $= (7 + 16.5 + 9) - 17.5 = 32.5 - 17.5$

Therefore, the area of ABCD = 15.

LOCUS

• PROBLEM 10-22

Write an equation for the locus of points equidistant from (3, 3) and (4, 4).



Solution: A perpendicular bisector is defined as the locus of points that are equidistant from the endpoints of a given segment. Therefore, the perpendicular bisector of the segment with endpoints (3, 3) and (4, 4) is the locus that is required.

FINDING THE EQUATION: A perpendicular bisector is a line. To find the equation of a line, we require a point on the line and the slope of the line. Because the perpendicular bisector includes the midpoint of the segment, we know that the midpoint between (3, 3) and (4, 4) is a point on the line. By the Midpoint Formula, we obtain

$$\left(\frac{3 + 4}{2}, \frac{3 + 4}{2}\right) = \left(\frac{7}{2}, \frac{7}{2}\right)$$

from the ends of the line segment ":

Draw the perpendicular bisector $\overleftrightarrow{CD}$ of $\overline{AB}$. Select a point P on $\overleftrightarrow{CD}$ and prove PA = PB.

This can be done by showing $\overline{PA}$ and $\overline{PB}$ are corresponding parts of ΔAPE and ΔBPE, proved congruent by the S.A.S. Postulate.

STATEMENT	REASON
1. $\overleftrightarrow{CD}$ is the $\perp$ bisector of $\overline{AB}$ and P is a point on $\overleftrightarrow{CD}$.	1. Given.
2. $\overline{AE} \cong \overline{EB}$	2. A bisector divides a line segment into two congruent parts.
3. $\measuredangle 1$ and $\measuredangle 2$ are right angles	3. Perpendicular lines intersect and form right angles.
4. $\measuredangle 1 \cong \measuredangle 2$	4. All right angles are congruent.
5. $\overline{PE} \cong \overline{PE}$	5. Reflexive property of congruence.
6. $\Delta APE \cong \Delta BPE$	6. S.A.S. $\cong$ S.A.S.
7. $\overline{PA} \cong \overline{PB}$	7. Corresponding sides of congruent triangles are congruent.
8. PA = PB	8. Congruent segments are equal in length.

(b) Proof of the converse "If a point is equidistant from the ends of a line segment, it is on the perpendicular bisector of the line segment."

Draw a point O, off $\overline{AB}$, such that PA = PB. If we let E be the midpoint of AB and can then show $\overleftrightarrow{PE}$, the bisector of $\overline{AB}$, is perpendicular to $\overline{AB}$, then the proof is done. We will show $\overleftrightarrow{PE} \perp \overline{AB}$ by proving congruence between adjacent angles $\measuredangle PEA$ and $\measuredangle PEB$. $\measuredangle PEA$ and $\measuredangle PEB$ will be shown to be corresponding angles of ΔAPE and ΔBPE, these proved congruent by the S.S.S. $\cong$ S.S.S. Postulate.

STATEMENT	REASON
1. Let E be the midpoint of $\overline{AB}$.	1. Every line segment has one and only one midpoint.

then we have

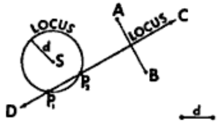
$$(2) \quad (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = d^2.$$

Assuming (x_2, y_2, z_2) and d are given, this is an equation for all points a distance d from (x_2, y_2, z_2). In other words, a sphere of radius d centered at (x_2, y_2, z_2).

By inspection we see that equation (2) is similar to equation (1). In fact, if we take $(x_2, y_2, z_2) = (1, 0, 0)$, and $d = 1$, the equations are identical. Hence, the locus described by equation (1) and the original equation is a sphere centered at (1, 0, 0) with radius d.

• PROBLEM 10-25

Locate the points that are a given distance, d, from a given point S, and are also equidistant from the ends of line segment $\overline{AB}$.



Solution: Conditions for two loci are given in this problem. The problem can be interpreted as one of locating the intersection points of these two loci.

The first condition, that the points of the locus be a given distance d from point S, describes a circle, the length of whose radius is d. This locus is the circle shown in the figure, with center at S.

The locus of points equidistant from the end of the line segment $\overline{AB}$ is the perpendicular bisector of $\overline{AB}$. In the figure, this is line $\overleftrightarrow{CD}$.

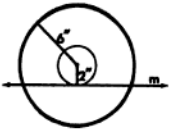
If d is less than the distance from S to $\overleftrightarrow{CD}$, then the circle locus and the locus given by $\overleftrightarrow{CD}$ do not intersect at any points and no points satisfy the two conditions.

If d is equal to the distance from S to $\overleftrightarrow{CD}$, then there is one point where both loci will intersect since $\overleftrightarrow{CD}$ will be the tangent of the circle centered at S with radius d. One point will satisfy both conditions.

If d is greater than the distance from S to $\overleftrightarrow{CD}$, then the locus $\overleftrightarrow{CD}$ is a secant to the circular locus and, accordingly, the two loci intersect at two points and these two points, P_1 and P_2, satisfy both conditions. This final case is exhibited in the figure.

• PROBLEM 10-26

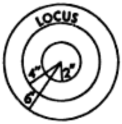
Two concentric circles have radii whose lengths are 2 in. and 6 in. Line m is drawn, in the accompanying figure, tangent to the smaller circle. (a) Describe fully the locus of points equidistant from the two circles. (b) Describe fully the locus of points at a given distance d from line m. (c) How many points are there which satisfy the conditions given in both parts (a) and (b) if: (1) d < 2 in.? (2) d = 2 in.? (3) d = 6 in.? (4) d > 6 in.?

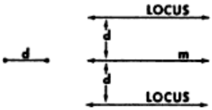


Solution: (a) The locus of points equidistant from the two concentric circles is a third concentric circle, the length of whose radius is 4 in.

The middle circle represents this locus.

(b) The locus of points at a given distance d from the given line m is a pair of lines, parallel to m, at the distance d from m. The 2 outer parallel lines shown comprise this locus.





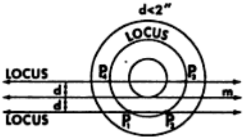


Fig. 1

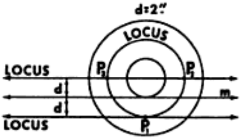


Fig. 2

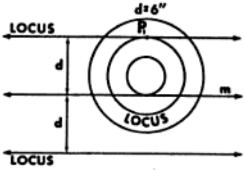


Fig. 3

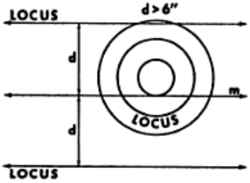


Fig. 4

(c) The accompanying figures, numbered 1 - 4, will be helpful in solving this problem.

(1) $d < 2$ in.: In this case, there are 4 points of intersection of the loci. In figure 1, they are P_1, P_2, P_3, P_4.

(2) $d = 2$ in.: As the distance between the parallel lines widens, the points of intersection approach each other. In case (1), both parallel lines were secants of the circular locus. However, when $d = 2$ in., one line becomes a tangent. Therefore, as fig. (2) shows, there are 3 points that satisfy both conditions: P_1, P_2 and P_3.

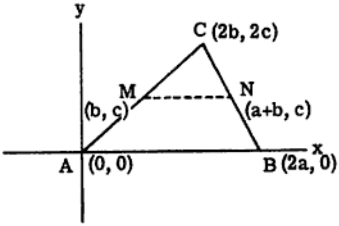
(3) $d = 6$ in.: The lowest of the lines in the parallel line locus, as shown in fig. (3), now lies completely outside of the circular locus and has no points of intersection with the latter. The upper line is tangent to the circle and provides the only point of intersection, P_1, that satisfies both conditions.

(d) $d > 6$ in.: The distance from line m to the outermost edge of the circular locus is 6 in. Therefore, when the parallel lines are over 6 in. away from line m, there will be no point which satisfies both conditions.

COORDINATE PROOFS

• PROBLEM 10-27

Prove that the segment formed by joining the midpoints of two sides of a triangle is parallel to the third side and has a length equal to half that of the third side.



Solution: There may be a certain amount of doubt that the figure above is completely general. If we prove the theorem for the above figure, have we really proven it for all triangles? It would seem that this is a very special triangle, indeed, that has one side on the X-axis and one vertex at the origin.

legs, i.e. showing that the Pythagorean Theorem holds.

The general formula for distance between two points will allow us to determine the required length. The formula tells us that for any two points (x_1, y_1) and (x_2, y_2) the length of the segment between them, d, is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \,.$$

Hence, since A = (2, - 2), B = (- 8, 4), and C = (5, 3), we can apply the formula to find:

$$AB = \sqrt{(-8-2)^2 + (4-(-2))^2} = \sqrt{(-10)^2 + (6)^2} = \sqrt{136}$$

$$BC = \sqrt{(5-(-8))^2 + (3-4)^2} = \sqrt{(13)^2 + (-1)^2} = \sqrt{170}$$

$$CA = \sqrt{(5-2)^2 + (3-(-2))^2} = \sqrt{(3)^2 + (5)^2} = \sqrt{34}$$

$\overline{BC}$ is the longest side of ΔABC and, if ΔABC is a right triangle, $\overline{BC}$ is the hypotenuse. Therefore, if $(BC)^2 = (AB)^2 + (CA)^2$, then ΔABC is a right triangle.

By substitution, we obtain

$$(\sqrt{170})^2 = (\sqrt{136})^2 + (\sqrt{34})^2$$

$$170 = 136 + 34 = 170.$$

Hence, ΔABC is a right triangle.

The area of a right triangle is equal to the one-half the product of the lengths of its legs.

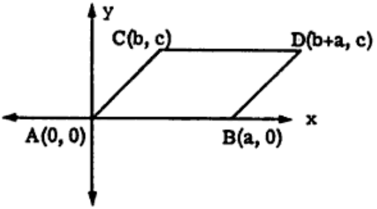
$$\text{Ergo, area of } \Delta ABC = \tfrac{1}{2}(\sqrt{136})(\sqrt{34}) = \tfrac{1}{2}\sqrt{4624}$$

$$= \tfrac{1}{2}\sqrt{16}\,\sqrt{289} = \tfrac{1}{2}(4)(17) = \tfrac{1}{2}(68).$$

Therefore, area of ΔABC is 34 sq. units.

• PROBLEM 10-29

Prove analytically that if the diagonals of a parallelogram are equal, the figure is a rectangle.



We draw a semicircle of radius r centered at the origin, as shown in the figure, the equation of the semicircle is $x^2 + y^2 = r^2$.

Let P_1 have the coordinates (x_1, y_1) and lie anywhere on the semicircle. Since r is the radius of the semicircle, the coordinates of A and B are $(-r, 0)$ and $(r, 0)$.

To prove that $\overline{P_1A} \perp \overline{P_1B}$ by analytical methods, we can show that the slopes of $\overline{P_1A}$ and $\overline{P_1B}$, m_1 and m_2 respectively, are in the relation $m_1m_2 = -1$.

By the definition of slope,

$$\text{Slope of } \overline{P_1A} = m_1 = \frac{y_1 - 0}{x_1 - (-r)} = \frac{y_1}{x_1 + r}$$

$$\text{Slope of } \overline{P_1B} = m_2 = \frac{y_1 - 0}{x_1 - r} = \frac{y_1}{x_1 - r}$$

$$\text{Hence, } m_1m_2 = \left(\frac{y_1}{x_1 - r}\right)\left(\frac{y_1}{x_1 + r}\right) = \frac{y_1^2}{x_1^2 - r^2}$$

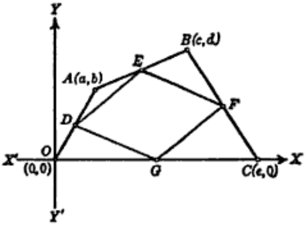
Since P_1 lies on a circle, the coordinates (x_1, y_1) satisfy the equation of the semicircle. Hence, $x_1^2 + y_1^2 = r^2$. It follows then that $y_1^2 = r^2 - x_1^2$, and $-y_1^2 = x_1^2 - r^2$.

$$\text{Therefore, by substitution, } m_1m_2 = \frac{y_1^2}{-y_1^2} = -1.$$

Hence, $\overline{P_1A} \perp \overline{P_1B}$ and $\sphericalangle AP_1B$ is a right angle.

• PROBLEM 10-31

Prove analytically that the lines joining the midpoints of the adjacent sides of any quadrilateral form a parallelogram.



Solution: In plane geometry we can prove a quadrilateral is a parallelogram by proving both pairs of opposite sides are parallel. This condition, in analytic geometry, amounts to proving that opposite sides have the same slope, since line segments with the same slope will always be parallel.

On the accompanying graph, an arbitrary quadrilateral has been positioned with one side on the x-axis and vertices A (a, b), B (c, d), C (e, 0), and O (0, 0) as shown. The midpoints of the sides are E, F, G, and D.

We want to prove that quadrilateral DEFG is a parallelogram. The slopes of the sides of this figure can readily be determined once we know the coordinates of D, E, F, and G.

Since these points are the midpoints of segments whose endpoints are known, by the midpoint formula, their coordinates can be determined.

D is the midpoint of $\overline{AO}$. Hence, $D = \left(\frac{a}{2}, \frac{b}{2}\right)$.

E is the midpoint of $\overline{AB}$. Hence $E = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

F is the midpoint of $\overline{BC}$. Hence, $F = \left(\frac{c+e}{2}, \frac{d}{2}\right)$.

G is the midpoint of $\overline{CO}$. Hence, $G = \left(\frac{e}{2}, 0\right)$.

Now we can proceed to calculate the slopes of the sides of DEFG. Recall, for any segment with endpoints (x_1, y_1) and (x_2, y_2) the slope equals $\frac{y_2 - y_1}{x_2 - x_1}$.

$$\text{Slope of } \overline{DE} = \frac{\frac{b}{2} - \frac{b+d}{2}}{\frac{a}{2} - \frac{a+c}{2}} = \frac{\frac{b-b-d}{2}}{\frac{a-a-c}{2}} = \frac{-d}{-c} = \frac{d}{c}$$

$$\text{Slope of } \overline{FG} = \frac{0 - \frac{d}{2}}{\frac{e}{2} - \frac{c+e}{2}} = \frac{-\frac{d}{2}}{\frac{e-c-e}{2}} = \frac{-d}{-c} = \frac{d}{c}$$

Hence, $\overline{DE} \parallel \overline{FG}$ since their slopes are equal.

$$\text{Slope of } \overline{EF} = \frac{\frac{b+d}{2} - \frac{d}{2}}{\frac{a+c}{2} - \frac{c+e}{2}} = \frac{\frac{b+d-d}{2}}{\frac{a+c-c-e}{2}} = \frac{b}{a-e}$$

$$\text{Slope of } \overline{DG} = \frac{\frac{b}{2} - 0}{\frac{a}{2} - \frac{e}{2}} = \frac{b}{a-e}$$

By similar reasoning, $\overline{EF} \parallel \overline{DG}$.

Therefore, DEFG is a parallelogram since all of its opposite sides are parallel.

CHAPTER 11

SOLID GEOMETRY

LOCUS

• PROBLEM 11-1

Graph $\{(x, y) : y \geq |x|\}$ where x and y are members of the set $\{-3, -2, -1, 0, 1, 2, 3\}$.

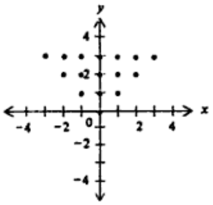


Figure 1

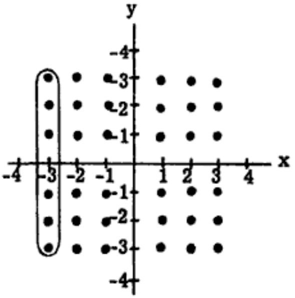


Figure 2

Solution: In this example, x and y are limited to a finite set of values. Therefore, there are a finite number of points. Thus for this graph, we need only to list all ordered pairs in the set, and plot each point, to complete the graph.

Each point on the graph must satisfy the equation

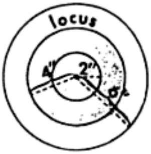
$$y \geq |x|.$$

To list all the points, we first choose a possible value of x, -3 for example, and find all the possible values of y such that $y \geq |-3|$, or $y \geq 3$. y is limited to the set $\{-3, -2, -1, 0, 1, 2, 3\}$. Therefore, the only possible value of y is 3. Point $(-3, 3)$ is, consequently, a point on the graph. Next, choose another value of x, -2 for example. For $y = 3$, it is true that $y \geq |-2|$. Also for $y = 2$, it is true that $y \geq |-2|$. Therefore, points $(-2, 2)$ and $(-2, 3)$ are on the graph. We continue choosing possible values of x and finding the corresponding y's such that $y \geq |x|$, until all x's have been chosen. When all x's have been tested, then our list of points is complete. The final list of points is $(-3, 3)$, $(-2, 3)$, $(-2, 2)$, $(-1, 3)$, $(-1, 2)$, $(-1, 1)$, $(0, 3)$, $(0, 2)$, $(0, 1)$, $(0, 0)$, $(1, 3)$, $(1, 2)$, $(1, 1)$, $(2, 3)$, $(2, 2)$, $(3, 3)$. The graph is shown in Fig. 1.

WHY THIS METHOD WORKS: If x and y must be members of the set $\{-3, -2, -1, 0, 1, 2, 3\}$, then the graph of all possible points in the set would be as in Fig. 2. By selecting $x = -3$, we were actually testing all the circled points in Fig. 2, and listing only those that satisfy the relation $y \geq |x|$. By repeating the procedure for every possible x, we see that all possible points are tested.

• PROBLEM 11-2

Describe the locus of points equidistant from two concentric spheres, the length of whose radii are 2 and 6.

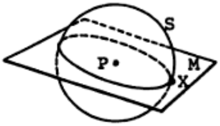


Solution: The locus of points that satisfy the given conditions must lie between the surfaces of the two spheres.

Since the original two spheres are concentric, a radius of the larger sphere will be concurrent with a radius of the smaller sphere. The distance between the two surfaces is the length of the larger radius minus the length of the smaller radius, i.e. $6 - 2 = 4$. One point in the locus must be at the midpoint of this distance, or 2 units from the inner spherical surface (or 4 units from the center). Repeating this process for every radius will result in our plotting a sphere of radius 4. Therefore, the locus described is a sphere of radius 4, with the same center as the given spheres.

• PROBLEM 11-3

In the figure, if plane M passes through the center, P, of the sphere S, prove that the intersection set of S and M is a great circle of S.



(c)

Solution: If S is a sphere of radius r, then any circle lying on S, with radius r, is a great circle of S. We will first show that the intersection set of S and M lies on sphere S. Then we show that the intersection set is a circle of radius r, and therefore is a great circle see figure.)

Since all the points of S lie on the sphere of radius r, the intersection set of S and M lies on the sphere. Hence, these points must also be members of S.

To show that the set of points is a great circle of P, note that (1) because the set of points are members of M, the set is collinear; (2) because the set of points are members of S, for every point X of the set of points, XP = radius of sphere; (3) P is on plane M.

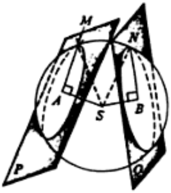
For a set of coplanar points, if there exists a point P coplanar with the set, such that the point is equidistant from every point of the set, then the set is a circle of center P and radius equal to the distance between each point and P. Therefore, the intersection set is a circle of radius XP = radius of sphere.

The intersection set is thus a circle on sphere S with radius equal to the radius of the sphere - in short, a great circle of S.

• PROBLEM 11-4

Given: Planes P and Q intersect sphere S in ⊙A and ⊙B, repectively; M is a point of ⊙A; N is a point of ⊙B: $\overline{AS} \cong \overline{BS}$.

Prove: ⊙A ≅ ⊙B.

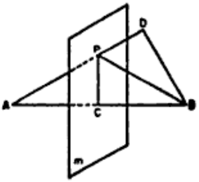


Solution: To prove two circles congruent, we must prove that their radii are congruent, in this case, $\overline{AM}$ and $\overline{BN}$. Whenever two line segments are to be equated, it is wise to look for congruent triangles. The congruent triangles for which we search must contain $\overline{AM}$ and $\overline{BN}$. In addition, since we are given that $\overline{AS} \cong \overline{BS}$, the triangles should also contain these segments. $\triangle AMS$ and $\triangle BNS$ satisfy these requirements. In proving $\triangle AMS \cong \triangle BNS$, we shall make use of three facts: (1) $\overline{AS} \cong \overline{BS}$ (given); (2) $\measuredangle$ MAS and $\measuredangle$ NBS are right angles (the line connecting the center of the sphere and the center of the intercepted circle is perpendicular to the intercepting plane); and (3) $\overline{MS} = \overline{NS}$ (all radii are equal).

Statements	Reasons
1. Planes P and Q intersect sphere S in ⊙A and ⊙B, respectively; M is a point of ⊙A; N is a point of ⊙B; $\overline{AS} \cong \overline{BS}$.	1. Given.
2. $\overline{MS} \cong \overline{NS}$.	2. All radii of a sphere are congruent. (Definition of a sphere.)
3. $\overline{AS} \perp$ plane P. $\overline{BS} \perp$ plane Q.	3. If a line contains the center of a sphere, and the center of a circle of intersection of the sphere with a plane not through the center of the sphere, then the line is perpendicular to the intersecting plane.
4. $\overline{AS} \perp \overline{AM}$. $\overline{BS} \perp \overline{BN}$.	4. A line perpendicular to a plane is perpendicular to every line in the plane that intersects the line.
5. $\measuredangle A$ and $\measuredangle B$ are right angles.	5. Definition of perpendicular lines.
6. $\triangle MAS$ and $\triangle NBS$ are right triangles.	6. Definition of right triangle.
7. $\triangle MAS \cong \triangle NBS$.	7. If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the two triangles are congruent.
8. $\overline{MA} \cong \overline{NB}$.	8. Correspoinding sides of congruent triangles are congruent.
9. ⊙A $\cong$ ⊙B.	9. Circles are congruent if and only if their radii are congruent.

• PROBLEM 11-5

Show that the locus of points equidistant from two given points is the plane perpendicular to the line segment joining them at their midpoint.



Solution: To prove that a certain set of points is a locus, we must show (1) all points in the set satisfy the conditions, and (2) all points that satisfy the conditions are contained in the set.

We satisfy the first condition by showing that any point P on m is equidistant from A and B (by showing $\Delta APC \cong \Delta BPC$). We satisfy the second condition by selecting a point outside the plane D on the B-side of the half-space and showing that $AD > DB$. Thus, any point not on the plane cannot be equidistant from both points.

Given: Points A and B, with plane m perpendicular to $\overline{AB}$ and bisecting $\overline{AB}$ at C.
To Prove: The locus of points equidistant from A and B is plane $m \perp \overline{AB}$ at its midpoint C.

Statements	Reasons
1. From P, any point in plane m, draw $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$.	1. Two points determine a line.
2. $\overline{AB} \perp \overline{PC}$.	2. If a line is $\perp$ to a plane, then it is $\perp$ to any line in the plane passing through the point of intersection.
3. $\overline{PC} \cong \overline{PC}$.	3. Reflexive Property.
4. $\overline{AC} \cong \overline{BC}$.	4. Definition of a midpoint.
5. $\angle PCA \cong \angle PCB$.	5. All right angles are congruent.
6. $\Delta PAC \cong \Delta PBC$.	6. The SAS Postulate.
7. $PA = PB$.	7. Corresponding parts of congruent triangles are equal in length.

which the lines lie are parallel. As such, the two lines of intersection will be parallel.

Given: Plane T intersects parallel planes P and Q.

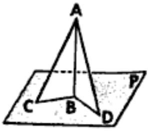
Prove: The intersection consists of two parallel lines.

Statements	Reasons
1. Plane T intersects parallel planes P and Q.	1. Given.
2. The intersection of T and P is line ℓ_1. The intersection of T and Q is line ℓ_2.	2. The intersection of two nonparallel, noncoincident planes is a line coplanar to both planes. (Plane T cannot be parallel or coincident to P or Q because it would contradict our assumption that T intersects P and Q).
3. ℓ_1 and ℓ_2 are coplanar.	3. From 2, we know that both ℓ_1 and ℓ_2 lie on plane T.
4. Either (1) ℓ_1 intersects ℓ_2 (2), or ℓ_1 does not intersect ℓ_2.	4. Either a statement or its negation is true.
5. Case (1) ℓ_1 intersects ℓ_2. Let r be a point on ℓ_1 and ℓ_2.	5. Definition of intersection of two lines.
6. r lies on P. r lies on Q.	6. If a line lies on a plane, then all points of the line are common to the plane.
7. There is no point r such that r lies on P and r lies on Q.	7. Parallel planes have no point in common.
8. ℓ_1 does not intersect ℓ_2.	8. Since statement (1) leads to a contradiction, the negation must be true.
9. $\ell_1 \parallel \ell_2$ or the intersection consists of two parallel lines.	9. Two coplanar lines that do not intersect are parallel.

4. Plane T intersects P at line $\overleftrightarrow{AC}$.	4. The intersection of two nonparallel, noncoincident planes is a line common to both planes.
5. $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$.	5. If a plane intersects two parallel planes, then it intersects them in two parallel lines.
6. $\overline{BA} \perp \overline{AC}$.	6. A line is perpendicular to a plane if, and only if, it is perpendicular to every line in the plane that passes through the point of intersection.
7. $\angle BAC$ is a right angle.	7. Perpendicular lines intersect to form right angles.
8. $m\angle BAC = 90°$.	8. All right angles measure 90°.
9. $m\angle BAC + m\angle ABD = 180$.	9. Interior angles of parallel lines on the same side of the transversal are supplementary.
10. $m\angle ABD = 90°$.	10. Subtraction Property of Equality.
11. $\overleftrightarrow{AB} \perp$ plane Q.	11. Since D could be any point in the plane, $\overleftrightarrow{AB}$ is perpendicular to every line $\overleftrightarrow{BD}$ in plane Q. By definition, $\overleftrightarrow{AB}$ is perpendicular to Q.

• PROBLEM 11-8

Given that $\overleftrightarrow{AB}$ is perpendicular to plane P, $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$ lie in plane P, and $\overline{BC} \cong \overline{BD}$, prove that $\overline{AC} \cong \overline{AD}$.



Solution: To prove that $\overline{AC} \cong \overline{AD}$, we must prove that the triangles in which these lines are corresponding sides, $\triangle ABC$ and $\triangle ABD$, are congruent. This will be done using the S.A.S. Postulate.

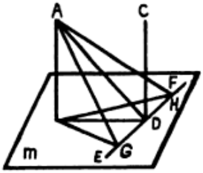
In order to prove that the included angles are congruent,

2. C is the intersection of line k with plane M.	2. The intersection of a line and a plane is a point.
3. $\overline{QL}$ is the intersection of planes M and P.	3. The intersection of two planes is a straight line.
4. Drop $\overline{CE} \perp \overline{QL}$ from point C.	4. A perpendicular may be drawn in a plane, from a point to a line.
5. Pass a plane through lines k and $\overleftrightarrow{CE}$ intersecting plane P in line $\overleftrightarrow{AB}$.	5. Two intersecting lines determine a plane.
6. Line k $\parallel$ $\overleftrightarrow{AB}$.	6. If a line is parallel to a plane, then it is parallel to the intersection of that plane with any plane containing the line.
7. Line k $\perp$ $\overline{CE}$.	7. A line perpendicular to a plane is perpendicular to every line in the plane passing through its foot.
8. $\overline{AB} \perp \overline{CE}$.	8. In a plane, if a line is perpendicular to one of two parallel lines, then it is perpendicular to the other.
9. $\overline{CE} \perp \overline{QL}$.	9. Construction.
10. $\overline{CE} \perp$ plane P.	10. If a line is perpendicular to two intersecting lines at their point of intersection, then it is perpendicular to the plane determined by the lines.
11. Plane M $\perp$ plane P.	11. If a line is perpendicular to a given plane, then every plane which contains this line is perependicular to the given plane.

• PROBLEM 11-10

Show that two lines perpendicular to the same plane are parallel.

Solution: In the accompanying figure, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular to plane m. We must prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. This can be done in two steps: first prove $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are coplanar and then show $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. To show $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are coplanar, we relate $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ to two other lines, $\overleftrightarrow{AD}$ and $\overleftrightarrow{BD}$.



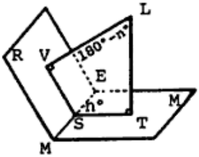
We use the theorem stating that all lines perpendicular to a line at a point are coplanar. We find a line such that $\overleftrightarrow{BD}$, $\overleftrightarrow{AD}$, and $\overleftrightarrow{CD}$ are all perpendicular to it at point D. Then $\overleftrightarrow{CD}$ is coplanar to the plane of $\overleftrightarrow{BD}$ and $\overleftrightarrow{AD}$. Since points A and B must lie in the plane of $\overleftrightarrow{BD}$ and $\overleftrightarrow{AD}$, so must the line $\overleftrightarrow{AB}$. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ both lie in the plane determined by $\overleftrightarrow{BD}$ and $\overleftrightarrow{AD}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{CD}$ are coplanar. To show that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, we show $\overleftrightarrow{AB} \perp \overleftrightarrow{BD}$ and $\overleftrightarrow{CD} \perp \overleftrightarrow{BD}$. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Statements	Reasons
1. Draw $\overline{BD}$ and $\overline{AD}$.	1. Two points determine a line.
2. In plane m, construct $\overline{EF} \perp \overline{BD}$.	2. At any point on a line, a ⊥ may be drawn.
3. On $\overline{EF}$, construct $\overline{HD} \cong \overline{DG}$ so that $\overline{BD}$ is ⊥ bisector of $\overline{HG}$.	3. Congruent line segments may be constructed on a line.
4. Draw $\overline{AH}$, $\overline{GA}$, $\overline{BH}$, and $\overline{BG}$.	4. Two points determine a line.
5. $\overline{BH} \cong \overline{BG}$.	5. A point on the ⊥ bisector of a line segment is equidistant from the ends of the line segment.
6. $\overline{AH} \cong \overline{AG}$.	6. If line segments drawn from a point in the ⊥ to a plane meet the plane at equal distances from the foot of the ⊥, then the line segments are equal. (This can be shown by proving $\triangle ABG \cong \triangle ABH$ by SAS.)
7. $\overline{AD} \perp \overline{HG}$.	7. The line determined by two points equidistant from the endpoints of the segments is the perpendicular bisector of the segment.
8. $\overline{CD} \perp \overline{EF}$.	8. If a line is ⊥ to a plane then it is ⊥ to any line in the plane passing through its foot.

14. $m\angle BVC + m\angle AVB > m\angle AVD + m\angle DVC$.	14. Substitution from Steps 12 and 3. Also, equals added to both sides of an inequality does not change the inequality.
15. $m\angle BVC + m\angle AVB > m\angle AVC$.	15. Substitution Postulate.

• PROBLEM 11-13

From a point within the dihedral angle formed by the intersection of two planes, perpendicular lines are drawn to each plane. (a) Prove that the plane determined by these perpendiculars is perpendicular to the edge of the dihedral angle. (b) If the number of degrees in the dihedral angle is represented by n, express in terms of n the number of degrees in the angle formed by the two perpendicular lines.



Solution: In the accompanying figure, point L is in the interior of dihedral angle R-$\overline{ME}$-N. $\overline{LT} \perp$ plane N and $\overline{LV} \perp$ plane R have been drawn.

We wish to prove plane LVT is $\perp \overline{ME}$, the edge of the dihedral angle. To do this remember that if a plane is perpendicular to two planes, then it is also perpendicular to their line of intersection. We will show that the plane LTV is perpendicular to planes N and R and thus is perpendicular to $\overline{ME}$.

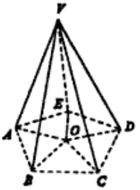
Statement	Reason
1. $\overline{LT} \perp$ plane N. $\overline{LV} \perp$ plane R.	1. Given.
2. $\overline{LV}$ and $\overline{LT}$ determine plane LVT.	2. Two intersecting lines determine a line.
3. Plane LTV $\perp$ plane N. Plane LTV $\perp$ plane R.	3. A plane containing a line perpendicular to a given plane is perpendicular to the given plane.
4. Plane LTV $\perp \overline{ME}$.	4. If a plane is $\perp$ to two planes, then it is $\perp$ to their line of intersection.

(b) We wish to find the measure of $\angle VLT$. This angle is part of a quadrilateral LVTS. The sum of the measures of the angles of a quadrilateral is 360°.

Since angles V and T each measure 90°, they sum to 180° and the remaining angles of LVTS, ∢VST and ∢VLT, sum to 360° - 180° = 180°. We are given m∢VST = n, therefore, we can conclude m∢VLT = 180° - n.

• PROBLEM 11-14

Show that the sum of the measures of the face angles of any convex polyhedral angle is less than 360°. An informal argument will suffice.



Solution: The only theorem about face angles we have learned thus far is that, for trihedral angles, the sum of any two face angles must be greater than the measure of the third. Here, we use the theorem to relate the supplements of the face angles to the interior angles of a convex polygon. Let a plane cut the edges of the polyhedral angle at points A, B, C, D, E. Let O be any point in the polygon ABCDE. Draw the segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, and $\overline{OE}$.

Call the n triangles with common vertex O the O-triangles. Call the n triangles with common vertex V (the triangles formed by the faces of the polyhedral angle) the V-triangles. Consider the base angles of the V triangles, ∢ABV, ∢BAV, ∢VBC, ∢VCB, etc., and the base angles of the O-triangles, ∢OAB, ∢OBA, ∢OBC, ∢OCB, etc. We show that (1) the sum of the base angles of the V triangles > the sum of the base angles of the O triangles; and, therefore, (2) the sum of the vertex angles (face angles) of V-triangles is less than the sum of the vertex angles (∢AOB, ∢BOC, ∢COD, etc.) of O-triangles. Since the vertex angles of O triangles sum to 360°, because a circle centered at O can be drawn, the sum of face angles must be less than 360°.

Given: Polyhedral angle ∢V-ABCDE...with n faces; the sum of the measures of the face angles equals S. Plane P intersects ∢V-ABCDE...at points A, B, C, D, E... .

Prove: S < 360.

Statements	Reasons
1. Polyhedral angle ∢V-ABCDE...with n faces;	1. Given.

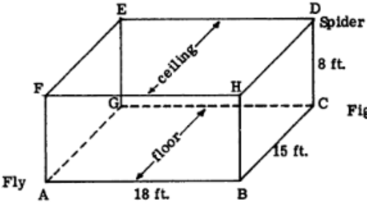


Figure 1

Solution: The elimination of confusing and irrelevant factors is the main difficulty in distance problems. For example, a person may drive in as straight a line as possible from New York to California, but because of mountains, valleys, and potholes in the road, the distance travelled will be greater than the straight line distance between New York and California. To determine the straight line distance, one must be able to eliminate the mountains and valleys.

In this problem, we are not concerned with the number of direction changes or transitions from wall to ceiling.

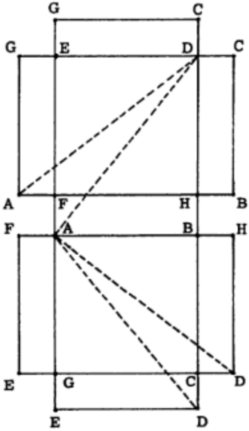


Figure 2

Therefore, we re-draw the picture, eliminate the wall-ceiling angles and wall-wall angles - this problem's equivalent of mountains and valleys. (See Figure 2.)

[Note that in flattening out the room, for completeness, three of the walls are represented twice. Thus, there are two locations D for the spider and two positions A for the fly.]

The problem reduces to finding the shortest distance between two points in a plane. Any one of the four dotted lines in Figure 2 is correct. Using the Pythagorean Theorem, we obtain:

$$\text{(i)} \quad AD^2 = AH^2 + DH^2$$

$$\text{(ii)} \quad AD^2 = (AF + FH)^2 + (15 \text{ ft.})^2$$

$$\text{(iii)} \quad AD^2 = (8 \text{ ft.} + 18 \text{ ft.})^2 + (15 \text{ ft.})^2$$

$$\text{(iv)} \quad AD^2 = 676 \text{ ft.}^2 + 225 \text{ ft.}^2$$

$$\text{(v)} \quad AD = \sqrt{901 \text{ ft.}^2} \cong 30 \text{ ft.}$$

The shortest path is illustrated in Figure 3.

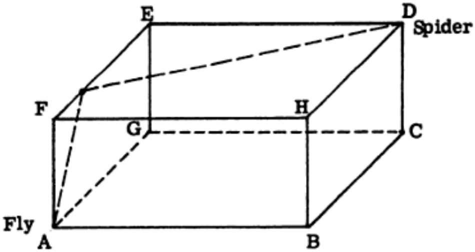
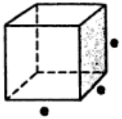


Figure 3

• PROBLEM 11-16

If the total surface area of the cube is 150 sq. in., find the length of an edge of the cube.



Solution: We have previously established that for a rectangular solid the total area, or T, is given by $T = S + 2B$, where S is the lateral area, and B is the base area. A cube is a specific case of a rectangular solid. Let e = the length of an edge. e is the same for all edges of a cube. If h is the cube's altitude, and p is the perimeter of the base, $S = hp$, For a cube, $h = e$ and $p = 4e$. Therefore,

$$S = e(4e) = 4e^2 \qquad B = e^2.$$

As such, $T = 4e^2 + 2(e^2) = 6e^2$.

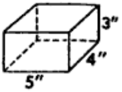
In this case, $T = 150 \text{ in.}^2$. By substitution,

$$T = 150 \text{ in.}^2 = 6e^2$$
$$25 \text{ in.}^2 = e^2$$
$$5 \text{ in.} = e.$$

Therefore: the length of an edge at the cube is 5 in.

• PROBLEM 11-17

Find the lateral area and the total surface area of a rectangular solid in which the dimensions of the base are 5 in. and 4 in. and the length of the altitude is 3 in.



Solution: The lateral area of a rectangular solid is the sum of the areas of its lateral faces, i.e. those perpendicular to the base. The total surface area is the lateral area plus the sum of the areas of the two bases.

Let the dimensions of the base be, in general, ℓ and w, and let the length of the altitude be h. Since opposite faces of a rectangular solid are of equal area, and all lateral faces have a height h, two opposing faces will have dimensions ℓ by h while two other opposing faces will have dimensions w by h. Based on the concept that the area of a rectangle is equal to the product of the lengths of its sides, it is correct to determine the lateral surface area by the equation

$$\ell h + wh + \ell h + wh = \text{Lateral Area.}$$

Factoring, we obtain lateral area $= h(\ell+w+\ell+w)$. Since $\ell + w + \ell + w = p$ (the perimeter of the base), then S (the lateral area of a rectangular solid) $= hp$. The total surface area includes the sum of the areas of the two bases plus the lateral area.

Total area = hp + 2(ℓw) or S + 2B, where B = ℓw.

In this problem, the perimeter of the base = (5+4+5+4) in. = 18 in. The altitude of the solid is h = 3 in. Lateral area of the solid = S = hp = $(3\times18)\text{in.}^2 = 54\ \text{in.}^2$. Area of the base = $(5\times4)\text{in.}^2 = (20)\text{in.}^2$ = B. Total surface area = S + 2(B) = $\left(54+2(20)\right)\text{in.}^2 = 94\ \text{in.}^2$. Therefore, Lateral area = 54 sq. in.; Total area = 94 sq. in.

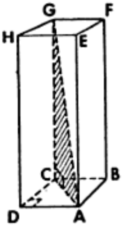
• PROBLEM 11-18

In the rectangular solid shown, $\overline{DA}$ = 4 in., $\overline{DC}$= 3, and $\overline{GC}$ = 12.

(a) Find the length of $\overline{CA}$, a diagonal of the base ABCD.

(b) Using the result found in part (a), find the length of $\overline{GA}$, a diagonal of the solid.

(c) If $\overline{DA} = \ell$, $\overline{DC}$ = w, and $\overline{GC}$ = h, represent the length of $\overline{GA}$ in terms of ℓ, w, and h.



Solution: All three parts of this problem will involve applying the Pythagorean Theorem in a solid geometry setting. Recall that the Pythagorean Theorem tells us that in a right triangle, the square of the length of the longest side, the hypotenuse, is equal to the sum of the squares of the lengths of the legs.

ABCD is a rectangle because the base of a rectangular solid is a rectangle. The diagonal of a rectangle partitions it into two right triangles. Knowing the length of two adjacent sides, $\overline{DC}$ and $\overline{DA}$, the length of the diagonal can be determined, as mentioned, by the Pythagorean Theorem.

In right ΔCDA, $(CA)^2 = (DA)^2 + (DC)^2$. By substitution, $(CA)^2 = (4)^2 + (3)^2 = 16 + 9 = 25$

CA $= \sqrt{25} = 5$. Therefore, CA = 5.

(b) Since the edge of a rectangular solid is perpendicular to the base of the solid, the edge is perpendicular to any line passing through its foot. (This follows from the definition of a line being perpendicular to a plane.) Therefore, $\overline{GC} \perp \overline{CA}$, and ΔGCA is a right triangle in which $\sphericalangle GCA$ is a right angle.

We want to find the length of $\overline{GA}$, the hypotenuse of ΔGCA, knowing the lengths of the other two legs. Once again, the Pythagorean Theorem is suggested.

In right ΔGCA, $(GA)^2 = (CA)^2 + (GC)^2$.

By substitution, $(GA)^2 = (5)^2 + (12)^2 = 25 + 144 = 169$,

$$GA = \sqrt{169} = 13.$$

Therefore, $\overline{GA}$, the diagonal of the solid, measures 13.

(c) In this part we must follow the same logic and reasoning as above. However, instead of substituting in numerical measurements, we are asked to be more general and substitute letters as variables for actual lengths.

To find GA, given $DA = \ell$, $DC = w$, and $GC = h$, we must first calculate CA.

As in part (a), $(CA)^2 = (DA)^2 + (DC)^2$.

Again, by substitution, $(CA)^2 = \ell^2 + w^2$. $CA = \sqrt{\ell^2 + w^2}$.

Now that we have represented $\overline{CA}$, we can proceed as in (b). Therefore, $(GA)^2 = (CA)^2 + (GC)^2$. Substitute to obtain

$$(GA)^2 = \left(\sqrt{\ell^2 + w^2}\right)^2 + h^2$$
$$(GA)^2 = \ell^2 + w^2 + h^2$$
$$GA = \sqrt{\ell^2 + w^2 + h^2}$$

Therefore, $GA = \sqrt{\ell^2 + w^2 + h^2}$.

Note: Given the length, ℓ, and width, w, of the base and the height, h, of the rectangular solid, the length of the diagonal of the solid, d, is given by $d = \sqrt{\ell^2 + w^2 + h^2}$.

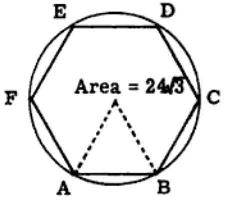
• PROBLEM 11-19

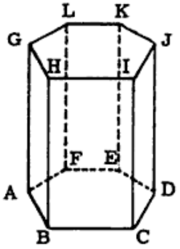
How many cubic feet are contained in a packing case which is a rectangular solid 4 ft. long, 3 ft. wide and $3\frac{1}{2}$ ft. high?

4. Quadrilaterals UQFJ, QRGF, TUJI... are parallelograms.	4. If both pairs of opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.
5. $\overline{UQ} \cong \overline{JF}$, $\overline{QR} \cong \overline{FG}$, $TU \cong IJ$...	5. Opposite sides of a parallelogram are congruent.
6. $\overline{TQ} \parallel \overline{IF}$, $\overline{UR} \parallel \overline{JG}$...	6. If a plane intersects two parallel planes, then they intersect in two parallel lines.
7. Quadrilaterals TIFQ, UJGR,... are parallel.	7. If both pairs of opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.
8. $\overline{TQ} \cong \overline{IF}$, $\overline{UR} \cong \overline{JG}$.	8. Opposite sides of a parallelogram are congruent.
9. $\triangle TUQ \cong \triangle IJF$, $\triangle UQR \cong \triangle JFG$...	9. The SSS Postulate.
10. $\sphericalangle TUQ \cong \sphericalangle IJF$, $\sphericalangle UQR \cong \sphericalangle JFG$...	10. Corresponding angles of congruent triangles are congruent.
11. Polygon QRSTU... $\cong$ base of P.	11. If the corresponding angles and the corresponding sides of two polygons are congruent, then the two polygons are congruent.

• PROBLEM 11-23

The base of a right prism is a regular hexagon with area $24\sqrt{3}$. If the lateral faces of the prism are squares, what is the lateral area?





__Solution__: The area of a prism = e $\cdot$ p where e is the length of the edge and p is the perimeter of a right section. Since this is a right prism, the bases themselves are right sections. Thus p = perimeter of hexagon = 6s (where s is the length of the side of the hexagon). Also, because each lateral face is a square, the edge e must equal the length of the side of the base s. Thus, area = $e \cdot p = s \cdot (6s) = 6s^2$.

We use the given area of the hexagon to solve for s. The area of a regular polygon is $\frac{1}{2}a \cdot p$ where a is the length of the apothem and p is perimeter. To solve for a and p would be lengthy and unnecessary. We note, instead, that the area of a regular hexagon is the sum of the areas of six equilateral triangles, each of side s. (In fact, this division of a regular polygon into congruent triangles is how the formula $\frac{1}{2}ap$ is usually proved.) Therefore,

Area of base = $24\sqrt{3}$ = 6 $\cdot$ (Area of equilateral triangle of

$$\text{side s}) = 6 \cdot \left(\frac{s^2\sqrt{3}}{4}\right) = \frac{3}{2}\sqrt{3}\ s^2$$

$$24\sqrt{3} = \frac{3}{2}\sqrt{3}\ s^2$$

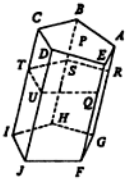
$$s^2 = \frac{24\sqrt{3}}{3/2\sqrt{3}} = \frac{2 \cdot 24}{3} = 16$$

$$s = \sqrt{16} = 4.$$

From above, we showed that the lateral area of the prism $= e \cdot p = s \cdot (6s) = 6s^2 = 6(4)^2 = 96$.

• PROBLEM 11-24

Show that the lateral area of a prism is equal to the product of the perimeter of a right section and the length of a lateral edge.



__Solution__: The lateral area of a prism equals the sum of the areas of the lateral faces, which, by definition, are parallelograms. The area of the nth parallelogram is $a_n b_n$ where a_n is the altitude and b_n is the lateral edge that forms the base. The lateral area of the entire prism equals $a_1b_1 + a_2b_2 + \dots + a_nb_n$. The lateral edges of a prism are congruent, $b_1 = b_2 = \dots b_n$. Therefore, the expression for lateral area can be expressed as $b_1(a_1 + a_2 + \dots a_n)$. We show that the altitudes form the sides of a right section of the prism.

Therefore, $a_1 + a_2 + \dots + a_n$ is the perimeter of the right section and the proof is complete.

Given: Prism P with right section QRSTU...; e the length of the lateral edge; Z the lateral area; w the perimeter of the right section.

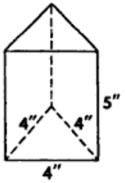
Prove: $Z = e \cdot w$.

Statements	Reasons
1. Prism P with right section QRSTU...; e the length of the lateral edge; Z the lateral area; w the perimeter of the right section.	1. Given
2. $\overline{DJ}$, $\overline{EF}$, $\overline{AG}$... $\perp$ QRSTU.	2. The right section of a prism is perpendicular to each lateral edge.
3. $\overline{DJ} \perp \overline{UQ}$, $\overline{EF} \perp \overline{QR}$...	3. A line that is perpendicular to a plane is perpendicular to each line in the plane that passes through the point of intersection.
4. $\overline{UQ}$, $\overline{QR}$, $\overline{RS}$...are altitudes of parallelograms DJFE, EFGA, BAGH, etc.	4. An altitude of a parallelogram is a line segment whose endpoints lie on opposite sides of the parallelogram and which is perpendicular to those sides.
5. Area of DJFE = $UQ \cdot EF$. Area of EFGA = $QR \cdot AG$.	5. The area of a parallelogram equals the length of the base times the length of the altitude drawn to the base.
6. Lateral Area of P = $UQ \cdot EF + QR \cdot AG + RS \cdot BH$...	6. The lateral area of a prism is the sum of the areas of the lateral faces.
7. $EF = AG = BH$...	7. The lateral edges of a prism are congruent.
8. Lateral Area of P = $EF \cdot (UQ + QR + RS\dots)$.	8. Substitution Postulate and Distributive Property.

9. $e = EF$.	9. Given.
10. $w = UQ + QR + RS \ldots$	10. The perimeter of a polygon is the sum of the lengths of its sides.
11. Lateral Area of $P = Z = e \cdot w$.	11. Substitution Postulate.

• PROBLEM 11-25

The base of a right prism, as shown in the figure, is an equilateral triangle, each of whose sides measures 4 units. The altitude of the prism measures 5 units. Find the volume of the prism. [Leave answer in radical form.]



Solution: We imagine the prism as a stack of equilateral triangles, congruent to the base of the prism. Let each of these triangles be one unit of measure thick. We can then calculate the area of the base, B, and multiply it by the number of bases needed to complete the height of the prism, h, to obtain the volume of the prism. Therefore, $V = Bh$. All prism volumes can be thought of in this way.

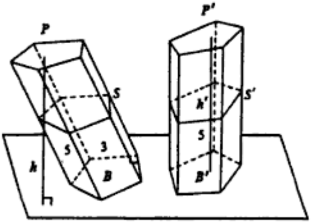
In this particular problem, the base is an equilateral triangle. Therefore $B = \frac{s^2\sqrt{3}}{4}$, where s is the length of a side of the base. By substitution, $B = \frac{(4)^2\sqrt{3}}{4} = 4\sqrt{3}$. Since the prism is 5 units high,

$$V = Bh = (4\sqrt{3})5 = 20\sqrt{3}.$$

Therefore, the volume of the prism is $20\sqrt{3}$ cu. units.

• PROBLEM 11-26

Show that two prisms have equal volumes if their bases have equal areas and their altitudes are equal.



Solution: In defining the volumes of polyhedral regions, three postulates are needed: (1) to each polyhedral region, there corresponds a unique positive number - in other words, every polyhedral region has one and only one volume; (2) the volume of a rectangular solid equals the product of its length, width, and height - this defines a standard from which other volume formulas can be derived; and (3) (Cavalieri's Principle) if two solid regions have equal altitudes, and if the sections, made by planes parallel to the base of each solid and at the same distance from each base, are always equal in area, then the volumes of the solid regions are equal - this provides a means of relating the volumes of all figures from the second postulate.

For this problem, we show that all sections cut by planes parallel to the bases of a prism are congruent to the bases, and thus, for prisms, Cavalieri's Postulate can be reduced to two requirements. For two prisms to have equal volumes, they must have (1) equal base areas, and (2) equal altitudes.

Given: Prisms P and P' with bases B and B' such that area of B equals area of B'; and altitudes h and h' such that h = h'.

Prove: Volume of P = Volume of P'.

Statements	Reasons
1. Prisms P and P' with bases B and B' such that area of B = area of B'; and altitudes h and h' such that h = h'.	1. Given.
2. Construct two planes parallel to the bases of P and P' making sections S and S' at equal distances d and d' from the bases B and B'.	2. In a given space, only one plane can be drawn a given distance from the given plane.
3. S ≅ B. S' ≅ B'.	3. Every section of a prism made by a plane parallel to the bases is congruent to the bases.

4. Area of S = Area of B Area of S' = Area of B'.	4. If two figures are congruent, then their areas are equal.
5. Area of S = Area of S'.	5. Transitivity Postulate.
6. Volume of P = Volume of P'.	6. If two solid regions have equal altitudes and if the sections made by planes parallel to the base of each solid and at the same distance from each base are always equal in area, then the volumes are equal. (Cavalieri's Principle.)

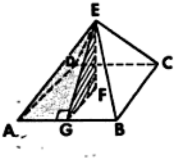
PYRAMIDS AND TETRAHEDRONS

• PROBLEM 11-27

In a regular square pyramid, the length of each side of the square base is 12 in., and the length of the altitude is 8 in.

(a) Find the length of the slant height of the pyramid.

(b) Find, in radical form, the length of the lateral edge of the pyramid.



Solution: Both parts of this example will involve an application of the Pythagorean Theorem in solid geometry. If h is the length of the hypotenuse, and a and b the lengths of the other legs of a right triangle, then, $h^2 = a^2 + b^2$.

(a) The slant height is the perpendicular from the vertex of the pyramid to any side of the base. Since each face of a regular pyramid is an isosceles triangle, the slant height bisects the base. To find the slant height, note that slant height $\overline{EG}$ is the hypotenuse of ΔEFG. The altitude of the pyramid, $\overline{EF}$, as shown in the figure, is perpendicular to the plane of the base ABCD, by definition. As such, $\overline{EF} \perp \overline{FG}$, because $\overline{FG}$ lies in the plane of the base and intersects $\overline{EF}$ at the latter's point of intersection with the base. The altitude must be drawn to the center of the base. Therefore, FG equals one-half the length of a side of the base.

dependent on the height of the pyramid and the distance between the polygon and the base. From an earlier theorem, we know that the ratio of the areas of similar polygons is proportional to the square of the ratio of similitude.

Given two sections parallel to the bases at equal distances and two pyramids with congruent altitudes, the ratio of similitudes is equal if the ratio of areas is equal. If we are further told that the bases of the pyramids have equal areas, then the area of the sections must also be equal.

Given: Pyramids V-ABC and U-DEFG, where $\overline{VK}$ is an altitude of V-ABC and $\overline{UL}$ an altitude of U-DEFG; $\overline{VK} \cong \overline{UL}$. Area of ΔABC = Area of DEFG; plane A'B'C' ∥ plane ABC and meets $\overline{VK}$ at K'; plane D'E'F'G' ∥ plane DEFG and meets $\overline{UL}$ in L'; $\overline{VK'} \cong \overline{UL'}$.

Prove: Area of ΔA'B'C' = Area of quadrilateral D'E'F'G'.

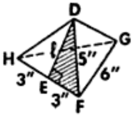
Statements	Reasons
1. (See above).	1. Given.
2. ΔA'B'C' ∿ ΔABC quadrilateral DEFG ∿ D'E'F'G'.	2. If a pyramid is cut by a plane parallel to its base, then the section is a polygon similar to the base.
3. The ratio of similitude of ΔA'B'C' and ΔABC = $\frac{A'B'}{AB}$. The ratio of similitude of quad D'E'F'G' and DEFG = $\frac{D'E'}{DE}$.	3. The ratio of similitude of two similar polygons is the proportion of the lengths of corresponding sides.
4. The ratio of similitude of ΔA'B'C' and ΔABC = $\frac{VK'}{VK}$. The ratio of similitude of quad D'E'F'G' and DEFG = $\frac{UG'}{UG}$.	4. If a pyramid is cut by a plane parallel to its base, then the ratio of the lengths of the altitudes is equal the ratio of similitude of the base and the section.
5. $\frac{\text{Area of A'B'C'}}{\text{Area of ABC}} = \left(\frac{VK'}{VK}\right)^2$. $\frac{\text{Area of quad D'E'F'G'}}{\text{Area of quad DEFG}} = \left(\frac{UG'}{UG}\right)^2$.	5. The ratio of the area of two similar polygons is the square of the ratio of similitude.
6. $\frac{VK'}{VK} = \frac{UG'}{UG}$	6. Division Property of Equality.

a) If a = 1, $s = \sqrt{3}$.

b) If a = 2, $s = 4\sqrt{3}$.

• PROBLEM 11-30

Find the lateral area and the total surface area of a regular triangular pyramid if each edge of the base measures 6 in. and each lateral edge of the pyramid measures 5 in. (Answer may be left in radical form. See figure.)



Solution: The lateral area is the sum of the areas of the lateral faces of the pyramid. The total surface area is the lateral area plus the area of the base.

Since the pyramid is regular, all the faces are congruent isosceles triangles, each of whose area is given by $\frac{1}{2}bh$, where b is the length of the base of the Δ, and h is the height. There are three such faces. We shall examine ΔDHF, shown in the figure.

The altitude, $\overline{DE}$, of ΔDHF, is perpendicular to $\overline{HF}$. Also, $\overline{DE}$ bisects $\overline{HF}$ because DHF is isosceles. Since HF = 6, EF = HE = 3. We need $\overline{DE}$ to find the area. By the Pythagorean Theorem, since ΔDEF is a right triangle,

$$(FD)^2 = (DE)^2 + (EF)^2$$

By substitution, $(5\text{ in.})^2 = (DE)^2 + (3\text{ in.})^2$

$$(DE)^2 = 25\text{ in.}^2 - 9\text{ in.}^2 = 16\text{ in.}^2$$

$$DE = 4\text{ in.}$$

Area of $\Delta DEF = \frac{1}{2}(DE)(HF)$

By substitution, Area of $\Delta DEF = \frac{1}{2}(4)(6)\text{in.}^2 = \frac{1}{2}(24)\text{ in.}^2$

$$= 12\text{ in.}^2$$

Lateral Area = 3(Area of ΔDEF) = $3(12)\text{in.}^2 = 36\text{ in.}^2$

The base is an equilateral triangle, since each edge of the base measures 6 in. Therefore,

$$\text{Base Area} = \frac{s^2\sqrt{3}}{4}$$

(the area of an equilateral triangle), where s is the length of an edge of the triangle. By substitution,

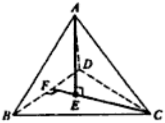
$$\text{Base Area} = \frac{(6)^2\sqrt{3}\ \text{in.}^2}{4} = 9\sqrt{3}\ \text{in.}^2$$

Total Surface Area = Lateral Area + Base Area = $(36 + 9\sqrt{3})\text{in.}^2$

Therefore, Lateral Area = 36 sq. in.; Total Surface Area = $(36 + 9\sqrt{3})$ sq. in.

• PROBLEM 11-31

Let ABCD be a regular tetrahedron with each edge of length 2. Let $\overline{AE}$ be perpendicular to the base, and assume that CE = $\frac{2}{3}$CF. (See figure). In ΔBCD, $\overline{CF} \perp \overline{BD}$. a) What is the length of CF? b) What is the length of CE? c) What is the length of AE? d) What is the base area of ΔBCD? e) What is the volume of the tetrahedron?



Solution:

a) We can prove that $\Delta CFB \cong \Delta CFD$ by the A.A.S. (angle angle side) Postulate in order to show DF = BF. Since we know DB, we can find DF. By using the Pythagorean Theorem we can determine CF. $\sphericalangle CFB \cong \sphericalangle CFD$ because $\overline{CF} \perp \overline{BD}$ and all right angles are congruent. Since all the edges of ABCD are of length 2, ΔDBC is equilateral. By definition, then, $\sphericalangle FBC \cong \sphericalangle FDC$, and $\overline{DC} \cong \overline{BC}$. By the A.A.S. Postulate, $\Delta CFB \cong \Delta CFD$, hence DF = BF. But DF + BF = DB. Thus,

$$DF = \frac{1}{2}DB.$$

Since DB = 2,

$$DF = 1.$$

Applying the Pythagorean Theorem to ΔDFC gives

$$(DF)^2 + (FC)^2 = (DC)^2$$

or $$FC = \sqrt{(DC)^2 - (DF)^2}.$$

Noting that DC = 2 and DF = 1

$$FC = \sqrt{4 - 1} = \sqrt{3}$$

b) By the statement of the problem,

$$CE = \frac{2}{3}CF.$$

From part (a), $CF = \sqrt{3}$. Hence,

$$CE = \frac{2}{3}\sqrt{3} = \frac{2\sqrt{3}}{3}.$$

c) Using the Pythagorean Theorem, for $\triangle AEC$,

$$(AE)^2 + (EC)^2 = (AC)^2$$

or

$$AE = \sqrt{(AC)^2 - (EC)^2}.$$

By realizing that AC is an edge of ABCD, we note that $AC = 2$. From part (b), $EC = \frac{2}{\sqrt{3}}$. Therefore,

$$AE = \sqrt{4 - 4/3}$$

$$AE = \sqrt{8/3}.$$

d) The area of ΔBCD is $\frac{1}{2}$ the product of its altitude (FC) and its base (DB). Hence,

$$\text{Area} = \frac{1}{2}(FC)(DB).$$

Noting that $FC = \sqrt{3}$ from part (a), and that $DB = 2$,

$$\text{Area} = (\tfrac{1}{2})(\sqrt{3})(2) = \sqrt{3}.$$

e) The volume of a tetrahedron of base area A and altitude h is

$$V = \frac{1}{3}Ah.$$

In our case $h = AE = \sqrt{8/3}$ from part (c). Hence,

$$V = (\tfrac{1}{3})(\sqrt{3})\left[\frac{\sqrt{8}}{\sqrt{3}}\right]$$

$$V = \frac{2\sqrt{2}}{3}.$$

• PROBLEM 11-32

The slant height of a regular pyramid A-BCD is 2. If the altitude is of length 1, find the volume.

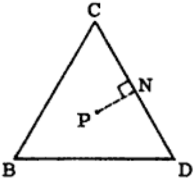


Figure 1

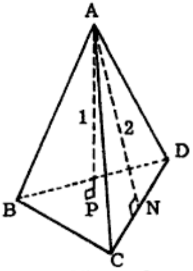


Figure 2

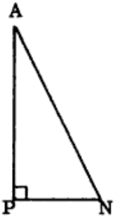


Figure 3

Solution: The volume of pyramid A-BCD equals one third the base area times the altitude. The altitude equals 1. Since this is a regular pyramid and only three points of its base are specified, the base must be a regular or equilateral triangle. The area of the base, therefore, equals $\frac{S^2\sqrt{3}}{4}$, where S is the length of a side. To find S, we show:

(1) $NC = \frac{1}{2}CD = \frac{1}{2}S$; (2) $BN = 3 \cdot PN$; (3) $S = BC = \sqrt{BN^2+NC^2}$

(1) Since A-BCD is a regular pyramid, its lateral edges are all congruent; therefore, face ADC is an isosceles triangle and, since the altitude drawn from the vertex angle of an isosceles triangle bisects the base, $CN = \frac{1}{2}CD = \frac{1}{2}S$.

(2) Since A-BCD is a regular pyramid, the altitude $\overline{AP}$ must intersect the base at the center of the base, or the point equidistant from all sides. But, by an earlier theorem, this point is also the point of concurrency of all angle bisectors of the triangle. Therefore, $\overline{BP}$ bisects $\sphericalangle B$. Consider equilateral triangle $\triangle BCD$ as an isosceles triangle of vertex B. Then, the angle bisector of the vertex angle bisects the base and therefore, angle bisector $\overline{BP}$ is also the median drawn to side $\overline{CD}$. By similar reasoning, considering $\triangle BCD$ as an isosceles triangle with C and D as vertex angles, we show that the angle bisectors of an equilateral triangle are the same as the medians of the triangle. Therefore, P, the point of concurrency of the angle bisectors, is the point of concurrency of the medians as well. Since the distance of the point of concurrency of medians from a given side is one third the length of the median, $PN = \frac{1}{3} \cdot BN$.

To find PN, note that PN is a leg in right triangle APN. By Pythagorean Theorem, $PN^2 = AN^2 - AP^2 = 2^2 - 1^2 = 3$. $PN = \sqrt{3}$.

(3) The median drawn to the base of an isosceles triangle is the perpendicular bisector of the base. Therefore, $\sphericalangle BNC$ is a right angle and $\triangle BNC$ is a right triangle. We wish to find the length of a side $\overline{BC}$. By Pythagorean Theorem,

$$= \frac{1}{3}h\left[b + \frac{\sqrt{b'}}{\sqrt{b} - \sqrt{b'}} \cdot \frac{(\sqrt{b})^2 - (\sqrt{b'})^2}{1}\right]$$

$$= \frac{1}{3}h\left[b + \frac{\sqrt{b'}}{\sqrt{b} - \sqrt{b'}} \quad \frac{(\sqrt{b} + \sqrt{b'})(\sqrt{b} - \sqrt{b'})}{1}\right]$$

$$= \frac{1}{3}h[b + \sqrt{b'}(\sqrt{b} + \sqrt{b'})]$$

$$= \frac{1}{3}h[b + b' + \sqrt{bb'}].$$

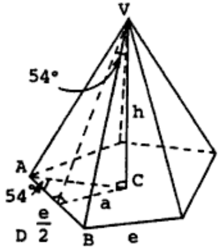
• PROBLEM 11-34

A regular pyramid has a pentagon for its base.

(a) Show that the area of the base is given by the formula $\beta = \frac{5}{4}e^2 \tan 54°$ where e = an edge of the base.

(b) If the slant height of the pyramid makes an angle of 54° with the altitude of the pyramid, show that the altitude is given by the formula $h = \frac{e}{2}$.

(c) Using the formulas of part a and b, write a formula for the volume of the pyramid in terms of the base edge e.



Solution:

(a) The area of a regular polygon A, whose perimeter is p and whose apothem is a, is given by the formula $A = \frac{1}{2}ap$. The perimeter of a regular n-gon equals n times the length of any one side. For this pentagon, p = 5e.

To find the length of the apothem a, draw the perpendicular from the center of the pentagon, C, to any side $\overline{AB}$, intersecting $\overline{AB}$ at point D. D is the midpoint of $\overline{AB}$. (To see this, remember that the center of a regular polygon is also the center of the circumscribing circle. Thus, $\overline{AC}$ and $\overline{BC}$ are radii of that circle. $\overline{AC} = \overline{BC}$ and $\triangle ABC$ is isosceles. $\overline{CD}$ is then the altitude drawn from the vertex angle of an isosceles triangle and therefore bisects the base.) Thus, $AD = \frac{1}{2}AB = \frac{e}{2}$. Furthermore, $m\angle CAD = 54°$. (To see this, note that all five central angles of the pentagon are congruent. Therefore, $m\angle ACB = \frac{360°}{5} = 72°$. Since $\triangle ABC$ is isosceles, $\angle CAD \cong \angle CBA$. By using the fact that the angle sum of a triangle is 180°, we have $m\angle CAD + m\angle CBA + m\angle ACB = 180°$, or $2m\angle CAD = 180° - 72°$ or $m\angle CAD = 54°$.)

In right $\triangle ACD$, then, a is the length of the leg opposite known $\angle CAD$. Leg AD is also known. The relationship between these three quantities is $\tan \angle CAD = \frac{\text{length of opposite leg}}{\text{length of adjacent leg}} = \frac{CD}{AD}$ or $\tan 54° = \frac{a}{e/2}$. Solving for a, we obtain $a = \frac{e}{2} \tan 54°$.

We have a and p. Thus, the base area equals $\frac{1}{2}ap = \frac{1}{2}(\frac{e}{2} \tan 54°)(5e) = \frac{5}{4}e^2 \tan 54°$.

(b) To find h, we construct another triangle, $\triangle VDC$. To show that $\triangle VDC$ is indeed a right triangle, we remember that the pyramid is regular. The perpendicular to the base of a regular pyramid passes through the center of the base. Since $\overline{VC}$ is the segment connecting the vertex to the center of the base, it is perpendicular to the base and $\angle VCD$ is a right angle. In $\triangle VDC$, we are given leg $\overline{DC} = a = \frac{e}{2} \tan 54°$. Furthermore, $\angle DVC$ is given to be 54°. Relating height $\overline{VC}$ to these quantities, we have

$$\tan 54° = \frac{\text{length of opposite leg}}{\text{length of adjacent leg}} = \frac{DC}{VC} = \frac{\frac{e}{2} \tan 54°}{\overline{VC}}.$$ Thus,

$$h = VC = \frac{\frac{e}{2} \tan 54°}{\tan 54°} = \frac{e}{2}.$$

(c) The volume of any pyramid is given by the formula $\frac{1}{3}bh$ where b = base area and h = height of the pyramid. From part (a), we know the base area to be $\frac{5}{4}e^2 \tan 54°$. From part (b), the height is $\frac{e}{2}$. Thus,

$$V = \frac{1}{3}bh = \frac{1}{3}(\frac{5}{4}e^2 \tan 54°)(\frac{e}{2}) = \frac{5}{24}e^3 \tan 54°.$$

c is is given by the formula $\frac{a+b+c-180}{720}$ s. s, the surface area of the sphere is $4\pi r^2 = 4\pi(9)^2$. Thus, the area of the triangle is $\frac{120+80+100-180}{720}(4\pi(9)^2) = \frac{120}{720}4\pi(9)^2 = \frac{2}{3}\pi(9)^2 = 54\pi$.

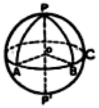
(d) The zone of a sphere is the region formed when two parallel planes intersect a sphere. That part of the sphere between the planes is called a zone and the distance between the planes is the altitude of the zone. The surface area of the zone is always equal to the lateral surface area of a cylinder of height equal to the altitude of the zone and base congruent to a great circle of the intercepted sphere. We are given that the surface area of the zone equals the area of the triangle. Thus,

$$(2\pi r)h = 54\pi.$$

Since $r = 9$, $h = \frac{54\pi}{2\pi(9)} = 3$ inches.

• PROBLEM 11-36

Show that if a point on a sphere is at a distance of a quadrant from each of two other points on the sphere, not the extremities of a diameter, then the point is a pole of the great circle passing through these points.



Solution: Let's begin by defining our terms. In a sphere of radius r, the great circle is a circle, all of whose points lie on the sphere, and which has a radius equal to the radius of the sphere. All great circles have a common center and that center is the center of the sphere. Most planar and linear measurements of the sphere involve the great circles in some way. The circumference of a sphere, for example, is the circumference of any great circle. In addition, the shortest distance between any two points of a sphere is found by drawing the great circle determined by the two points. The part of the circumference between the two points is the geodesic - or path of shortest distance. Examples of great circles are the equator of the earth and the prime meridian.

A quadrant of a sphere is defined to be one-fourth the great circle. It is a sector of a circle of radius r, and the measure of its central angle is $\frac{360}{4} = 90°$. The distance of a quadrant is the arc of the quadrant. The North Pole and any point on the equator are a distance of a quadrant away from each other.

For every great circle, there are two points on the sphere, one on each side of the great circle, that are a quadrant away from the circumference of the great circle. These are the poles of the great circle. The segment with the two poles as endpoints is called the axis and is a diameter of the sphere. Thus, the axis intersects the great circle at the center of the sphere. In addition, this diameter is perpendicular to the plane of the great sphere.

In the accompanying figure, sphere O has great circle ABC as a great circle. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants.

We must show that P is a pole of ⊙ABC. To show that a point of a sphere is a pole a great circle, it is sufficient to show that the axis of the point $\overline{POP'}$ is perpendicular to the plane of the great circle. We show $\overline{PO} \perp \overline{AO}$ and $\overline{PO} \perp \overline{OB}$. Since $\overline{PO}$ is perpendicular to two lines in plane AOB, $\overline{PO} \perp$ plane of the great circle.

Given: Points P, A, and B on the sphere O; $\overset{\frown}{AB}$ of a great circle ABC; and the quadrants $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$.

To Prove: P is a pole of ⊙ABC.

Statements	Reasons
1. Draw $\overline{AO}$, $\overline{BO}$ and diameter $\overline{POP'}$.	1. Two points determine a line; a diameter is a line segment passing through the center and having its endpoints on the sphere.
2. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants.	2. Given.
3. ∢POA and ∢POB are right angles; or $\overline{PO} \perp \overline{OA}$, $\overline{PO} \perp \overline{OB}$.	3. The central angle of a quadrant is a right angle.
4. $\overline{PO} \perp$ plane ABC (or $\overline{PP'} \perp$ ABC).	4. If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane of the lines.
5. PP' is the axis of ⊙ABC.	5. The axis of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.
6. P is a pole of ⊙ABC.	6. The extremities of the axis are the poles of the circle.

• PROBLEM 11-37

The most remote spot of ocean, the point at 48°30' S and 125°30' W in the Pacific, is 1660 miles from the nearest land Pitcairn Island. Approximate the area of this oceanic expanse as (1) a circle; (2) a zone of a sphere. Compare the two. (The radius of the earth is 3960 mi.)

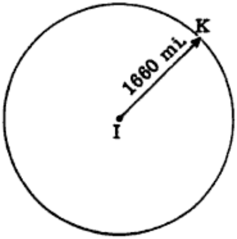


Figure 1

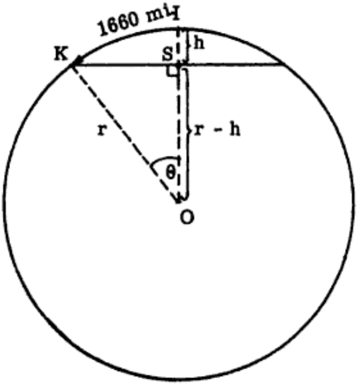


Figure 2

Solution:

(1) The area of a circle with radius r equals πr^2. Therefore, the area of the watery expanse is $\pi(1660 \text{ mi.})^2 = 2.76 \times 10^6 \pi \text{ mi.}^2$

(2) The area of a zone is given by the formula $2\pi rh$. To find h, we first find θ. With θ, we can find $\overline{OS}$. Since h is the radius minus $\overline{OS}$, we can find h if we know θ.

To find θ, remember that the arc length and central angle measure are proportional

$$\text{(i)} \quad \frac{\theta}{360} = \frac{m\widehat{IK}}{\text{circum. of earth}}.$$

The circumference of the earth (of radius 3960 miles) is $2\pi r = 2\pi(3{,}960 \text{ mi.})$. IK is given to be 1660 mi.

$$\text{(ii)} \quad \frac{\theta}{360} = \frac{1660 \text{ mi.}}{2\pi(3960 \text{ mi.})}$$

$$\text{(iii)} \quad \theta = 360 \frac{1660}{2\pi \cdot 3{,}960} = \frac{830}{11\pi} = 24°.$$

To find $\overline{OS}$, note that in right ΔKSO:

$$\text{(iv)} \quad \cos\theta = \frac{OS}{KO} = \frac{r-h}{r} = \frac{3{,}960 \text{ mi} - h}{3960}.$$

Consulting the trigonometric tables, we find cos 24° = .914. Substituting this value in and multiplying both sides by 3,960, we obtain:

$$\text{(v)} \quad (.914)(3{,}960) = 3{,}960 - h$$

$$\text{(vi)} \quad h = 3{,}960 - (.914)(3{,}960) = 3{,}960(1-.914)$$

$$= 3{,}960(.086) = 341 \text{ mi.}$$

With a value for h, we can now solve for the area of the zone:

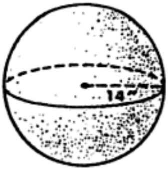
$$\text{(vii) Area} = 2\pi rh = 2\pi(3960 \text{ mi.})(341 \text{ mi.}) =$$

$$2.7 \times 10^6 \text{ mi.}^2$$

The zone approximation is smaller than the planar approximation by 2%. In either case, the area covered by the water is greater than any single nation.

• PROBLEM 11-38

Find the surface area of a sphere whose radius measures 14 in. [Use $\pi = \frac{22}{7}$] (See figure.)



Solution: The surface area of a sphere is, by postulate, equal to four times the area of one of its great circles.

Therefore, $S = 4\pi r^2$, where r is the radius of a great circle of the sphere. Since the radius of a great circle of a sphere is equal to the radius of the sphere itself, r = 14 in. By substitution,

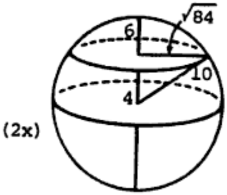
$$S = 4\left(\frac{22}{7}\right)(14)^2 \text{ in.}^2$$

$$S = 2464 \text{ in.}^2$$

Therefore, area of the sphere = 2464 sq. in.

The ratio of the altitude of a zone to the diameter of the sphere on which it is drawn is 1:5. The area of the zone is 80π.

a) Find the area of the sphere.
b) If one of the bases of the zone is a great circle, find the area of the other base.



Solution:

(a) To find the area of the sphere we use the formula $S = 4\pi r^2$. To use the formula, though, we must solve for r. We solve for r using the given zone area. We know that the zone area and r are related by the formula,

$$A_{zone} = (\text{circumference of the great circle}) \cdot (\text{zone height})$$

$$80\pi = 2\pi r \cdot h.$$

This equation can be solved for r if we can eliminate the second variable h. From the given, though, we know that $h:d = 1:5$ or $h = \frac{1}{5}d$. Since the diameter d is twice the radius, we have $h = \frac{1}{5}(2r) = \frac{2}{5}r$. Substituting this result in the zone area formula, we have

$$80\pi = 2\pi r \cdot (\tfrac{2}{5}r) = \tfrac{4}{5}\pi r^2$$

$$r^2 = 80\pi \cdot (\tfrac{5}{4\pi}) = 100$$

$$r = 10.$$

Thus the radius of the sphere is 10 and its area equals $4\pi r^2 = 4\pi(10)^2 = 400\pi$.

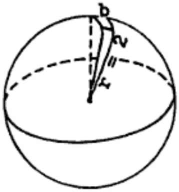
(b) To find the area of the smaller base, we use the formula πr_1^2, where r_1 is the radius of the smaller base, To find r_1, consider the line drawn from the center of the sphere perpendicular to the plane of the circle. By an earlier theorem, we know that this line intersects the circle at its center. Thus, a right triangle is formed with hypotenuse equal to the radius of the sphere, one leg equal to the altitude of the zone and one leg equal to the radius of the smaller base. The radius of the sphere equals 10 (see Part a). The altitude of the zone can be found by the relationship used in Part a, namely $h = \frac{2}{5}r$, where r is the radius of the sphere. So $h = \frac{2}{5}(10) = 4$. Then, by the Pythagorean Theorem, we have

$$\begin{aligned}(\text{radius of circle})^2 &= (\text{radius of sphere})^2 - (\text{altitude of zone})^2 \\ &= 10^2 - 4^2 \\ &= 84.\end{aligned}$$

Since the area of the circle = $\pi(\text{radius of circle})^2$. we have $\pi(84) = 84\pi$.

• PROBLEM 11-40

Find the volume of sphere of radius 2.



Solution: The volume of a sphere equals $\frac{4}{3}\pi r^3$. To see this, consider any polygon on the surface of the sphere. Draw the radii of the sphere to the vertices. The resulting solid resembles a pyramid with altitude roughly equal to the radius and a base that is not a plane surface but slightly spherical. By choosing polygons of smaller and smaller areas, the base becomes less curved and more like a plane figure; and, thus, the volume of the pyramidlike figure approaches $\frac{1}{3}r \cdot b$ where b is the area of the base polygon and r, the radius of the circle, is approximately the altitude. Suppose we divide the sphere into n pyramidlike figures. Then, if $b_1, b_2, \ldots, b_n$ are the areas of each base, the volume of the sphere equals the sum of the volumes of the n pyramids:

$$V = \frac{1}{3}rb_1 + \frac{1}{3}rb_2 + \ldots + \frac{1}{3}rb_n.$$

Factoring out $\frac{1}{3}r$, we obtain

Solution: In Figure 1, sphere P of radius r is circumscribed by regular tetrahedron ABCD. Sphere P is tangent with face ABC at point E; face ABD at F; face ACD at G; and face BCD at H. s is the length of the edge of tetrahedron ABCD. We wish to find the volume V of ABCD in terms of the radius r. From an earlier proof, we know that the volume of a tetrahedron is $\frac{\sqrt{2}}{12}s^3$ where s is the length of the edge. Therefore, we find s in terms of r; substitute in the tetrahedron volume formula; and the problem is solved.

We will find the relationship of the edge AB = s and the sphere radius PE = r. We do this by considering triangle AMH in Figure 2, formed by the slant height $\overline{AM}$ and the perpendicular from the vertex A to the base $\overline{AH}$. We can find AM and MH. Furthermore, from an earlier proof, we know that (1) since P, the sphere center, is equidistant from the faces of the regular pyramid (the distances to the faces from an inscribed sphere are radii. Since all radii are congruent, the center is equidistant), and since all points equidistant from the faces of a regular pyramid lie on the perpendicular from the vertex, then P must lie on the perpendicular $\overline{AH}$, or $\overleftrightarrow{APH}$; and (2) the perpendicular ($\overline{PE}$) to the face of a regular pyramid (A-BCD...) from a point (P) on the vertex perpendicular ($\overleftrightarrow{AH}$) intersects the face (ABC) on its slant height ($\overleftrightarrow{AM}$). Therefore, $\overleftrightarrow{AEM}$.

Combining all these facts, we construct Figure 3. Note that ∡AEP and ∡AHM are right angles (thus, ∢AEP ≅ ∢AHM) and ∢A ≅ ∢A. Therefore, ΔAEP ∼ ΔAHM by the A-A Similarity Theorem. Because the sides of similar triangles are proportional, we have

$$\text{(i)} \quad \frac{EP}{AP} = \frac{HM}{AM}.$$

Because EP is a radius, EP = r. To find AM, note that $\overline{AM}$ is a leg of right ΔABM. (ΔABM is a right triangle because a slant height of a face is always ⊥ to the base.) We know that AB = s. Furthermore, because face ΔABC is an equilateral triangle (every face of a regular tetrahedron is an equilateral triangle) perpendicular $\overline{AM}$ bisects base $\overline{BC}$, and $BM = \frac{1}{2}BC = \frac{1}{2}s$. By the Pythagorean Theorem, $AM^2 = BA^2 - BM^2 = s^2 - \left(\frac{1}{2}s\right)^2 = \frac{3}{4}s^2$, or $AM = \frac{\sqrt{3}}{2}s$.

To find MH, remember that the perpendicular from the vertex of a regular pyramid intersects the base polygon at its center. Therefore, H is the center of the base equilateral ΔBCD — that is H is the point of concurrency of the altitudes, angle bisectors, and medians. (Note in an equilateral triangle, these are the same three lines.) Because H is the concurrency point of the medians, it divides the median DM such that $HD = \frac{2}{3}DM$ or $HM = \frac{1}{3}DM$. By considering right ΔBDM, we show by the Pythagorean Theorem that $DM^2 = BD^2 - BM^2 = s^2 - \left(\frac{1}{2}s\right)^2$

$= \frac{3}{4}s^2$ or $DM = \frac{\sqrt{3}}{2}s$. Since $HM = \frac{1}{3}DM$, then $HM = \frac{1}{3}\left(\frac{\sqrt{3}}{2}s\right) = \frac{\sqrt{3}}{6}s$.

Returning to the equation (i) and substituting in our values, we obtain:

$$\text{(ii)} \quad \frac{r}{AP} = \frac{\frac{\sqrt{3}}{6}s}{\frac{\sqrt{3}}{2}s} \text{ or } r = \frac{\frac{\sqrt{3}}{6}s}{\frac{\sqrt{3}}{2}s}AP = \frac{1}{3}AP.$$

Refer back to Figure 3. We have discovered that $r = \frac{1}{3}AP$ or $AP = 3r$. Since $\overline{PH}$ is a radius of the sphere, $PH = r$, and thus $AH = AP + PH = 3r + r = 4r$. In right ΔAMH, we have hypotenuse AM expressible in terms of s, leg MH expressible in terms of s, and leg AH expressible in terms of r. By using Pythagorean Theorem, we can relate s to r.

$$\text{(iii)} \quad AM^2 = MH^2 + AH^2$$

$$\text{(iv)} \quad \left(\frac{\sqrt{3}}{2}s\right)^2 = \left(\frac{\sqrt{3}}{6}s\right)^2 + (4r)^2$$

$$\text{(v)} \quad \frac{3}{4}s^2 = \frac{1}{12}s^2 + 16r^2.$$

$$\text{(vi)} \quad 16r^2 = \frac{3}{4}s^2 - \frac{1}{12}s^2 = \left(\frac{9}{12} - \frac{1}{12}\right)s^2 = \frac{2}{3}s^2$$

$$\text{(vii)} \quad \frac{3}{2} \cdot 16r^2 = s^2$$

$$\text{(viii)} \quad 24r^2 = s^2$$

$$\text{(ix)} \quad s = \sqrt{24}r = 2\sqrt{6}r.$$

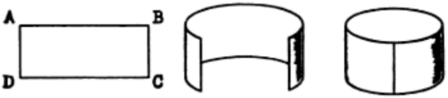
Now, we can use the volume formula.

$$\text{(x)} \quad V = \frac{\sqrt{2}}{12}s^3 = \frac{\sqrt{2}}{12}(2\sqrt{6}r)^3 = 8\sqrt{3}r^3.$$

CYLINDERS

• PROBLEM 11-42

A right circular cylinder has a base whose diameter is 7 and height is 10. What is the surface area of the cylinder, not including the bases?



Solution: From the figure, we see that calculating the required surface area is equivalent to calculating the area of the rectangle shown. The area of the rectangle, K, is

$$K = (AB)(BC).$$

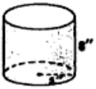
But BC is the height of the cylinder, and BC = 10. AB is equivalent to the circumference of the base of the cylinder. If the base has diameter d, the circumference of the base is πd, which equals 7π in our case. Hence, AB = 7π. Therefore,

$$K = (7\pi)(10) = 70\pi$$

is the required surface area.

• PROBLEM 11-43

Find, in terms of π, the lateral area and the total surface area of a right circular cylinder if the radius of its base measures 5 in. and its altitude measures 8 in. (See figure).



Solution: The lateral face of a right circular cylinder is actually a rectangle, whose base length (b) is the circumference of the base of the cylinder, and whose height (h) is the height of the cylinder. Therefore, the lateral area is equal to bh. In the case of a right circular cylinder, b = $2\pi r$, the circumference of the base, where r is the base radius.

$$\text{Lateral Area} = 2\pi rh.$$

By substitution, Lateral Area = $2\pi(5 \text{ in.})(8 \text{ in.}) = 80\pi \text{ in.}^2$ The total surface area equals the lateral area plus the sum of the area of the two bases.

Area of one Base = Area of a circle = πr^2

$$= \pi(5)^2 \text{ in.}^2 = 25\pi \text{ in.}^2$$

Area of the two Bases = $2(25\pi \text{ in.}^2) = 50\pi \text{ in.}^2$

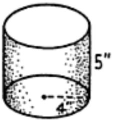
Total Surface Area = $(80\pi + 50\pi)\text{in.}^2 = 130\pi \text{ in.}^2$

Therefore, Lateral area = 80π sq. in.

Total Surface area = 130π sq. in.

• PROBLEM 11-44

Find, in terms of π, the volume of a right circular cylinder if the radius of its base measures 4 in. and its altitude measures 5 in.



Solution: If we picture the base of the cylinder as having a depth of one unit of measure, we can then calculate the volume by determining the area of the base and multiplying it by the height of the cylinder. In effect, this multiplication amounts to stacking the bases up to the height of the cylinder.

Area of the base = πr^2, where r is the radius of the circular base. Therefore, the volume is given by

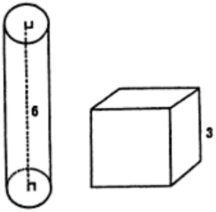
$$V = \pi r^2 h.$$

By substitution, $V = \pi(4)^2 5 \text{ in.}^3 = 80\pi \text{in.}^3$.

Therefore, volume of the cylinder = $80\pi \text{ in.}^3$.

• PROBLEM 11-45

Find the volume of a right circular cylinder of height 6 if it has the same lateral surface area as a cube of edge 3.



Solution: The volume of a cylinder equals $\pi r^2 h$. The height, h, equals 6. The radius can be found by relating the lateral surface area of the cube and the cylinder. The lateral surface area of the cylinder equals (base perimeter) · (height) = $2\pi rh$. The lateral surface area of the cube which has no bases equals $4s^2$. By equating these two expressions, we obtain

$$2\pi rh = 4s^2.$$

By substitution, $\qquad 2\pi r(6) = 4(3)^2$

$$r = \frac{4(3)^2}{2\pi(6)} = \frac{36}{2\pi(6)} = \frac{3}{\pi}.$$

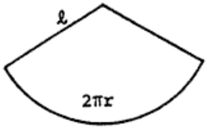
Therefore,

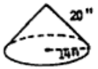
$$\begin{aligned} \text{Volume of cylinder} &= (\text{Area of base}) \cdot (\text{height}) \\ &= \pi r^2 \cdot h \\ &= \pi\left(\frac{3}{\pi}\right)^2 \cdot 6 \\ &= \frac{54}{\pi}. \end{aligned}$$

CONES

• PROBLEM 11-46

Find the lateral area and the total area of a right circular cone in which the radius measures 14 in. and the slant height measures 20 in. [Use $\pi = \frac{22}{7}$].





Solution: The lateral surface of a right circular cone, with base radius r, when laid flat, appears to be a sector of a large circle. As such, its area is equal to the area of that sector.

The radius of the sector is equal to the slant height, ℓ, of the cone. The arc length intercepted is equal to the circumference of the base of the cone, $2\pi r$.

Area of the Sector

$$= (\text{Area of entire circle})\left(\frac{\text{Central Angle measure}}{\text{Measure of full circle}}\right).$$

Let θ = Central angle measure.

$$\text{Area of the Sector} = \cancel{\pi}\ell^2\left(\frac{\theta}{2\cancel{\pi}}\right) = \ell^2\,\frac{\theta}{2}.$$

All arcs and angles are measured in radians. In radians,

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{2\pi r}{\ell}.$$

Therefore, area of the sector = $\ell^2(\frac{1}{2})(\frac{2\pi r}{\ell})$.

Area of the sector = $\pi r\ell$ = Lateral area of the cone.

Hence, by substitution,

Lateral Area of the cone = $\pi(14)(20)\text{in.}^2 = (\frac{22}{7})(14)(20)\text{in.}^2$

$$= 880 \text{ in.}^2$$

Total Area = Lateral area + Base area

$$= \pi r\ell + \pi r^2$$

$$= 880 \text{ in.}^2 + \frac{22}{7}(14)^2 \text{ in.}^2$$

$$= (880 + 616)\text{in.}^2 = 1496 \text{ in.}^2$$

Therefore, Lateral Area = 880 sq. in.

Total Area = 1496 sq. in.

• PROBLEM 11-47

The flat pattern in the accompanying diagram is used to make a lampshade. It is the lateral surface of the frustum of a right circular cone.

The two concentric circles have radii of 12 and 4 inches and $m\sphericalangle AOB = 60°$. The minor sector AOB and the remainder of the interior of the circle are removed and discarded. $\overline{AC}$ is fastened to $\overline{BD}$ with no space or overlap.

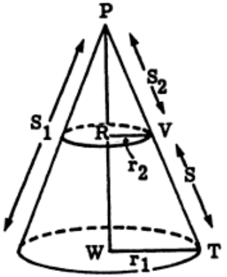
(a) Find the radius of each of the two bases of the lampshade.

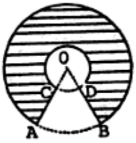
(b) Find the area of the outer surface of the lampshade. [Leave answer in terms of π.]

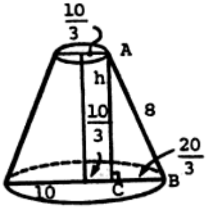
(c) Find the altitude of the lampshade. [Leave in radical form.]

Solution:

(a) The smaller base of the lampshade will be formed out of the portion of the smaller circle remaining after the 60° sector has been removed. Since 60° is $\frac{1}{6}$ of the total measure of the circle, the smaller base of the lampshade will be a circle whose circumference will be $\frac{5}{6}$ of the original small circle.







Since the circumference is a linear function of the radius, the ratio of the circumferences of any two circles will be equal to the ratio of the radii. Thus, the radius $\overline{OC}$ of the smaller circle will be $\frac{5}{6}$ of the original smaller radius of 4. Therefore, the radius of the smaller base of the lampshade is $\frac{5}{6}(4)$ or $\frac{10}{3}$ inches.

By similar logic, the radius of the larger base is $\frac{5}{6}$ of the original large radius of 12. The larger base has a radius of $\frac{5}{6}(12)$ or 10 inches.

Therefore, the radii of the bases of the lampshade are $\frac{10}{3}$ in. and 10 in.

(b) The area of the outer surface of the lampshade has the form of the surface area of a frustum of a cone. Therefore, its area can be found by first looking at a frustum in general.

Assume that a frustum of a cone is formed with base radii r_1 and r_2 and slant height s. The entire cone has slant height s_1 and the upper cone is left with slant height s_2.

The lateral area of the entire cone is, in general, π(base radius)(slant height). In this case, $L_1 = \pi r_1 s_1$.

The lateral area of the upper cone is $L_2 = \pi r_2 s_2$. Hence, the lateral area of the frustum is $L = L_1 - L_2 = \pi(r_1 s_1 - r_2 s_2)$. Since we are only given data about the frustum, we don't know s_1 and s_2 and, as such, must try to eliminate them.

We note that ΔPWT is a right triangle and since $r_2 \parallel r_1$

(this is the case in all frustums of right circular cones). $\Delta PRV \sim \Delta PWT$, by the AA Similarity Theorem. Therefore,

$$\frac{s_2}{s_1} = \frac{r_2}{r_1} \quad \text{and} \quad s_2 = \frac{s_1 r_2}{r_1}.$$

Now, by substituting into the equation for L, we obtain

$$L = \pi\left(r_1 s_1 - \frac{s_1 r_2^2}{r_1}\right) = \frac{\pi}{r_1}(r_1^2 s_1 - s_1 r_2^2) = \frac{\pi s_1}{r_1}(r_1^2 - r_2^2)$$

$$= \frac{\pi s_1}{r_1}(r_1^2 - r_2^2) = \frac{\pi s_1}{r_1}(r_1 - r_2)(r_1 + r_2)$$

$$= \left(\pi s_1 - \frac{\pi s_1 r_2}{r_1}\right)(r_1 + r_2).$$

Substituting $s_2 = \frac{s_1 r_2}{r_1}$ into the last equation,

$$L = \pi(s_1 - s_2)(r_1 + r_2).$$

We see that $s = s_1 - s_2$. Therefore,

$$L = \pi s(r_1 + r_2).$$

In our lampshade problem we know $r_1 = 10$, $r_2 = \frac{10}{3}$ and $s = CA = DB =$ the difference between the original two radii, 12 - 4, or 8.

By substitution,

$$L = \pi(8)(10 + \tfrac{10}{3}) = \pi(8)(\tfrac{40}{3}) = \tfrac{320}{3}\pi.$$

Therefore, the area of the outer surface is $\frac{320\pi}{3}$ sq. in.

(c) We can find the altitude, AC, by reconqizing that ΔABC is a right triangle and then applying the Pythagorean Theorem. The hypotenuse of ΔABC is 8 in., since it corresponds to the slant height of the frustum.

The top radius is $\frac{10}{3}$ in and the bottom radius is 10 in. By projecting the top radius onto the bottom we see that $CB = 10 - \frac{10}{3} = \frac{20}{3}$.

Let h = the length of the altitude. Then,

$$h^2 = 8^2 - (\tfrac{20}{3})^2 = 64 - \tfrac{400}{9} = \tfrac{576 - 400}{9} = \tfrac{176}{9}$$

c). Figure (B) shows the region obtained by cutting the cone of figure (A) along $\overline{AC}$. This region has the same area as the surface area of the cone. The area of the region can be calculated from the following proportion

$$\frac{\text{Area(region)}}{\text{Area(circle O)}} = \frac{\text{length } \widehat{ABA'}}{\text{length of circle O}}$$

$$\text{Area(region)} = \frac{(\pi s^2)(2\pi r)}{(2\pi s)}.$$

Here we have used the fact that the length of $\widehat{ABA'}$ is $2\pi r$, since $\widehat{ABA'}$ is the boundary of the circular base of the cone. Hence,

$$\text{Area(region)} = \pi rs.$$

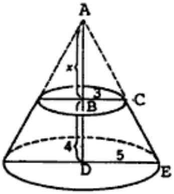
In our case, $s = 5$ ft. and $r = 3$ ft. Then

$$\text{Area(region)} = 15\pi \text{ ft.}^2 = 47.12 \text{ ft.}^2.$$

The surface area of the cone is therefore 47.12 ft.^{2}.

• PROBLEM 11-49

A frustum of a solid cone is the portion of solid cone remaining when the solid cone above a given section is removed. (See figure.) (a). Using the properties of similar triangles, what can we say about $x/3$ and $(x + 4)/5$? (b). Find the height, $x + 4$, of the large cone. (c) Find the volume of the small solid cone. (d) Find the volume of the large solid cone. (e) Find the volume of the frustum. (f) Taking $r_1 = 3$, $r_2 = 5$, and $h = 4$, compute $V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$. Compare this with part (e).



Solution:

(a) We will prove that $\Delta ABC \sim \Delta ADE$, by the A.A. (angle-angle) Similarity Theorem. We will then be able to set up a proportion between $\overline{AB}$, $\overline{AD}$, $\overline{BC}$, and $\overline{DE}$, thereby finding a relationship between $x/3$ and $(x + 4)/5$.

First, $\sphericalangle A$ is common to both $\triangle ABC$ and $\triangle ADE$. Therefore, $\sphericalangle BAC \cong \sphericalangle DAE$. Furthermore, the plane containing $\overline{BC}$ is parallel to the base of the cone. Hence, $\sphericalangle ACB \cong \sphericalangle AED$, since they are corresponding angles of the parallel segments $\overline{BC}$ and $\overline{DE}$. In conclusion, $\triangle ABC \sim \triangle ADE$, and we may write

$$\frac{AB}{AD} = \frac{BC}{DE}.$$

Since $AB = x$, $AD = x + 4$, $BC = 3$ and $DE = 5$, the last equation may be rewritten as

$$\frac{x}{x+4} = \frac{3}{5}$$

or

$$5x = 3x + 12$$

or

$$x = 6.$$

Hence, $\frac{x+4}{5} = 2$ and $\frac{x}{3} = 2$ (i.e., $\frac{x+4}{5} = \frac{x}{3}$).

(b) The height of the large cone, $x + 4$, is, from part (a), 10.

(c) The volume of the small cone is equal to $\frac{\pi}{3}$ times the product of its height (x) and the square of the length of the radius of its base (3^2). Hence,

$$V_S = (\tfrac{\pi}{3})(x)(3^2)$$

$$V_S = (\tfrac{\pi}{3})(6)(9) = 56.52.$$

(d) The volume of the large solid cone is equal to $\frac{\pi}{3}$ times the product of its height ($x + 4$) and the square of the length of the radius of its base (5^2). Hence,

$$V_L = (\tfrac{\pi}{3})(x+4)(5^2)$$

$$V_L = (\tfrac{\pi}{3})(10)(25) = 261.67.$$

(e) The volume of the large cone, minus the volume of the small cone, equals the volume of the frustum, (V_F). Therefore,

$$V_F = V_L - V_S = 261.67 - 56.52$$

$$V_F = 205.15.$$

(f) If $r_1 = 3$, $r_2 = 5$, and $h = 4$,

$$V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$$

$$V = (\frac{\pi}{3})(4)((3)^2 + (3)(5) + (5)^2)$$

$$V = (\frac{\pi}{3})(4)(9 + 15 + 25)$$

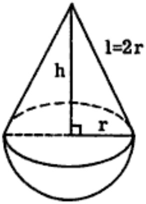
$$V = (\frac{4\pi}{3})(49)$$

$$V = 205.15.$$

Note that this conforms to the volume of the frustum as calculated in part (e).

• PROBLEM 11-50

In the accompanying figure, a right circular cone is constructed on the base of a hemisphere. The surface of the hemisphere is equal to the lateral surface of the cone. Show that the volume V of the solid formed can be found by the formula $V = \frac{1}{3}\pi r^3(2 + \sqrt{3})$.



Solution: The volume of the solid formed equals the volume of the hemisphere plus the volume of the right circular cone. The volume of a hemisphere is half the volume of a sphere. Thus, $V_{hemi.} = \frac{1}{2}(\frac{4}{3}\pi r^3) = \frac{2}{3}\pi r^3$. The volume of a cone of base radius r and height h equals $\frac{1}{3}\pi r^2 h$. (Note in this case that the base of the cone is the great circle of the hemisphere. Thus, r, the base radius of the cone, equals r, the radius of the hemisphere.) The desired volume, V, of the solid is, therefore, $V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h = \frac{\pi}{3}r^2(2r + h)$.

The problem, though, asks us to express the volume totally in terms of r, not r and h.

Consider the right triangle formed by altitude h, base radius r and slant height ℓ. By the Pythagorean Theorem, $h^2 = \ell^2 - r^2$. r is known; and ℓ and r are related by the surface area formulas. Thus, ℓ can be expressed in terms of r and the results substituted in the Pythagorean equation above to express h in terms of r.

Surface Area of hemisphere = $2\pi r^2$.

Lateral Area of cone = $\pi r\ell$.

By the given, the two surface areas are equal. Then, $2\pi r^2 = \pi r\ell$ or $\ell = 2r$.

Using the Pythagorean equation, we have $h^2 = (2r)^2 - r^2 = 4r^2 - r^2 = 3r^2$. Thus, $h = \sqrt{3}r$.

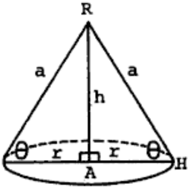
We substitute this in our expression for V and obtain $V = \frac{\pi}{3}r^2(2r + \sqrt{3}r) = \frac{\pi}{3}r^3(2 + \sqrt{3})$.

• PROBLEM 11-51

An isosceles triangle, each of whose base angles is θ and whose legs are a, is rotated through 180°, using as an axis its altitude to the base.

(a) Find the volume V of the resulting solid, in terms of θ and a.

(b) Using the formula found in part a, find V, to the nearest integer if a = 5.2 and θ = 27°. (Use $\pi = 3.14$)



Solution:

(a) The resulting figure is a right circular cone. Its volume, V, is given by $V = \frac{1}{3}\pi r^2 h$ where h is the altitude of the cone and r is the radius of the base. First, we express r and h in terms of the given angle, θ, and the given length, a.

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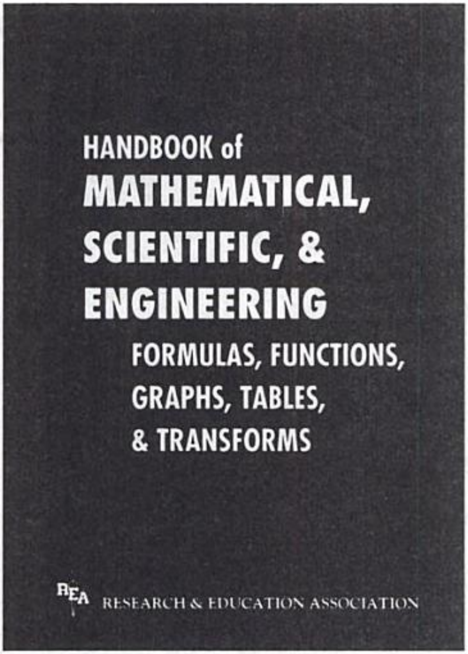

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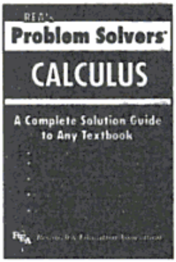

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